

**STUDY OF VARIOUS QUANTUM  
INFORMATION THEORETIC RESOURCES  
AND THEIR APPLICATIONS**

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To  
*My Parents...*

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## ABSTRACT

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Information processing in the domain of the quantum world is already a vastly enriched area of research both from theoretical as well as practical perspectives due to the growing and enormous demand of daily life. During the last three decades, it has been well-established that various quantum systems can act as resource for several information processing, communication and cryptographic tasks and are shown to be advantageous over their classical counterparts. The present thesis is devoted to theoretical studies of the detection and application of various quantum information theoretic resources which do not have any classical analogue. Among several useful quantum resources, in the present thesis we explore the detection of EPR-steering, non-Markovianity (NM) and sequential detection of genuine tripartite entanglement. Considering the uncertainty bound on the sum of variances of two incompatible observables, we derive the sum-steering inequality. Our steering criteria not only yields the optimum steering range for two-qubit Werner states in the two-measurement and two-outcome scenario but also provides a tighter steering condition to reveal the steerability of continuous-variable states. Next, considering the sequential measurement scenario, we explore the possibilities of sequential detection of a single genuinely tripartite entangled state. Since, creating or distributing genuinely entangled states among distant observers is very much difficult in a real experiment, it is important to explore the possibilities of using a single genuinely multipartite state multiple times. In the later part of the thesis, a formalism for detection of NM has been presented through the violation of uncertainty relations. The last part of the thesis covers the application of indefinite causal order, another important quantum resource. By introducing a task called *random-receiver quantum communication*, we establish the supremacy of indefinite causal order over that of the superposition in path degrees of freedom. Finally, in the conclusion chapter we summarize our main findings with speculation on future directions.

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## PUBLICATIONS

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### Publications relevant to the Thesis:

1. "Tighter Einstein-Podolsky-Rosen steering inequality based on the sum uncertainty relation", A. G. Maity, S. Datta and A. S. Majumdar, Phys. Rev. A, **96**, 052326 (2017).
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3. "Detecting non-Markovianity via uncertainty relations", A. G. Maity, S. Bhattacharya and A. S. Majumdar, J. Phys. A: Math. and Theo. **53**, 175301 (2020).
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### Additional publications during the Ph.D. thesis but not forming part of it:

1. "Genuinely nonlocal product bases: Classification and entanglement-assisted discrimination", S. Rout, A. G. Maity, A. Mukherjee, S. Halder and M. Banik, Phys. Rev. A, **100**, 032321 (2019).
2. "Local state discrimination and ordering of multipartite entangled states", S. Rout, A. G. Maity, A. Mukherjee, S. Halder and M. Banik, arXiv:1910.14308 (2019).
3. "Sequential detection of genuine tripartite entanglement in semi-device-independent scenarios", S. Gupta, A. G. Maity, D. Das, A. Roy and A. S. Majumdar, Phys. Rev. A, **103**, 022421 (2021).

4. *"Self-testing of binary measurements requiring neither entanglement nor any dimensional restriction"*, A. G. Maity, S. Mal, C. Jebarathinam and A. S. Majumdar, Phys. Rev. A, 103, 062604 (2021).
5. *"Unbounded pairs of observers can achieve quantum advantage in random access codes with a single pair of qubits"*, D. Das, A. Ghosal, S. Kanjilal, A. G. Maity and A. Roy, arXiv : 2101.01227 (2021).

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## FOREWARD

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Information theory, as its name suggests, is the mathematical study of various tasks such as detection, quantification, storage, transmission and utilization of information. Although the field was initially introduced around 1920 by the works of Harry Nyquist [1] and Ralph Hartley [2], modern information theory was initiated with the seminal work of Claude Shannon in 1948 [3]. In the paper ‘A Mathematical Theory of Communication’, Shannon established the theoretical foundations of almost all present day communication tasks by introducing the qualitative and quantitative model of communication. By proposing his famous source coding theorem, Shannon established the ultimate theoretical bound of possible data compression, and provided the operational meaning of the *entropy*, a key measure of information. *Entropy* quantifies the amount of uncertainty present in the value of a random variable. For example, specifying the definite outcome of an unbiased coin toss which has two equally likely outcomes gives less information (lower entropy) than pointing out the definite outcome of a rolling die where there are six equally likely outcomes. It was Shannon who first introduced the term *bit*, the most fundamental unit of information which determines the amount of information needed to specify the outcome definitely out of two equally probable events of a random process. Shannon however considered the information storage and information carrying devices as classical.

During the last four decades, with the increasing demand of daily life, much of the information and communication technologies have been developed gradually from their classical to quantum counterparts due to the fact that usage of quantum systems for the purpose of information processing provide advantages over its classical counterpart. Moreover, in

the quantum domain some practically relevant tasks are possible that cannot be obtained through classical devices – secure cryptography and communication are such examples. It is *entanglement* and several other quantum correlations which act as fuel behind all these phenomena. The essence of *entanglement* follows from the idea of inseparability of composite quantum systems. Consider two particles that have been interacted in the past but now are kept at a distance and are not able to interact with one another. Although they do not interact, they are still called to be *entangled* if the state of the joint system cannot be written as a product of the states of individual subsystems. The term, *entanglement* thus refers to the quantum correlations between spatially separated physical systems. The idea started long ago with the seminal work of Einstein, Podolsky and Rosen (EPR)[4] in 1935 through a meta-physical argument called local-realism. Considering a position-momentum correlated state of two spatially separated particles (electron-positron pair), they concluded that nonlocality is an artefact of the incompleteness of quantum mechanics. The nomenclature ‘*entanglement*’ was however coined by Schrödinger [5, 6] in a follow-up to the EPR contribution to describe spatially separated but correlated particles. The term ‘*steering*’, was also introduced to describe how the choice of a measurement settings on one side could affect the state on the other. Much later, it was J. S. Bell [7, 8] who proposed an experimentally testable criterion of incompatibility of quantum mechanics with that of local-realistic description of classical physics. Incorporating the fundamental assumptions of the EPR paper, Bell formulated certain inequalities known as ‘Bell inequalities’ and showed that the correlations, obtained from some local measurements performed on two spatially separated particles have a local-realistic description only if these correlations obey those inequalities. However, there exist certain entangled states that violate those inequalities and therefore quantum mechanics cannot be embraced within the framework of local realism. This astonishing feature of quantum correlations is coined as quantum nonlocality and has enormous applications. There are many experimental realisations of the violation of Bell inequalities confirming the inconsistency of local-realism with quantum mechanics [9, 10].

Bell inequalities pointed out to us that entanglement is the necessary physical resource which gives rise to nonlocal quantum correlations and is fundamentally new and different from the classical resources. The relation between entanglement and nonlocality is in gen-

eral very complex but it was Werner who provided an explicit example from two-qubit states showing entanglement is only a necessary condition for quantum nonlocality but is not sufficient. Later, Wiseman *et al.* [11] formulated a unified information theoretic description of quantum correlations manifested through entanglement, EPR-steering and Bell nonlocality in terms of information theoretic tasks. As a corollary, a strict hierarchy has also been established between these three types of correlation, *viz.*, EPR-steering lies between Bell nonlocality and entanglement, with the latter being the weakest. The idea of quantum correlations has also been extended to the multipartite systems.

Apart from those foundational concepts, in recent years information theoretic aspects of Entanglement and other quantum correlations have also been investigated. The path started with the revolutionary idea of teleportation by Bennet *et al.* [12]. Subsequently, superdense coding [13], quantum computation [14], remote state preparation [15, 16], quantum key distribution [17, 18], randomness certification [19], random access code [20, 21], winning of nonlocal games [22] etc. have also been explored leading to the interesting and secure information processing, cryptographic and communication tasks which otherwise were not possible in classical physics. In order to utilize those resources, at first it is important to characterize and identify them. Among several useful quantum correlations which act as resources, the present study focuses on the detection of EPR-steering and sequential detection of genuine tripartite entanglement.

On the other hand, the theory of open quantum systems suggests that as a result of decoherence, the system monotonically relaxes to the thermal equilibrium or more generally to a non-equilibrium steady state. The one way information flow, characterized by the monotonic relaxation towards the stationary states is a direct consequence of the Born-Markov approximation [23]. For very large stationary environments, Born-Markov approximation leads to the CP (Complete Positive)-divisibility of the dynamics. However, in realistic situations where system-environment coupling is not sufficiently weak, Born-Markov approximation is not valid and as a consequence the CP-divisibility may break down, leading to the observation of non-Markovian backflow of information [24, 25] from environment to the system. In recent years, it has been shown that non-Markovian backflow of information can also act as a resource in several information processing tasks such as perfect teleportation with mixed

states [26], increasing the capacity of a quantum channel [27], efficient entanglement distribution [28], efficient work extraction [29] and so on. The present thesis also aims to detect the signature of the information-backflow in an experimentally feasible way.

In recent times there is significant research interest to investigate physically relevant situations where the causal ordering between events is itself controlled by quantum theory and hence may become indefinite [30, 31]. This plays an important role to build a more fundamental theory of nature that will include both the properties of dynamic causal structure of general relativity and the probabilistic nature of quantum theory [32, 33]. One interesting primitive of such indefinite causal structure is the *quantum SWITCH*, which has already been realized experimentally in different photonic setups [34, 35].

In any communication scenario, unwanted noise is always unavoidable and hence, the main research aim in quantum communication theory is to find the optimal use of noisy quantum channels for reliable information transfer. Such a channel can be most generally modeled by a completely positive trace preserving map on the input quantum system. One of the surprising results in the quantum scenario is the super-additivity phenomena – two completely useless channels can be combined in a way that becomes useful, a fact which is impossible for classical channels. Very recently it has been found that such an amazing activation phenomena can also be obtained from the perspective of indefinite causal order. There are several other interesting applications of the *quantum SWITCH*, enabling quantum advantages in various tasks, such as testing properties of quantum channels [36], winning nonlocal games [30, 31], and reducing quantum communication complexity [37], enhancing classical and quantum information capacity of noisy quantum channels [38, 39]. In the present study, we present other new primitives of indefinite causal order and show their novelty over superposition of path by designing a new quantum communication task. This may have far reaching applications in the emerging field of quantum networks and distant communication.

*Plan of the thesis* :- The thesis is arranged in the following manner.

*Chapter 2* :- Chapter 2 revisits the mathematical preliminaries and physical requisites necessary to follow the contents of this thesis. This includes a brief introduction of the known

protocols in quantum information science, like Bell nonlocality, EPR paradox, EPR-steering, Entanglement, Genuine entanglement, open quantum system, indefinite causal order and *quantum SWITCH*.

*Chapter 3* :- In this chapter, we derive a criterion for the detection of EPR-steering, an important resource for various information processing and cryptographic tasks. Our derived steering inequality is based on the sum of inferred variances pertaining to two observables of bipartite systems and is shown to be advantageous over Reid inequality, based on Heisenberg uncertainty relation as well as entropic steering inequality, based on entropic uncertainty relation.

*Chapter 4* :- In this chapter we discuss about the sequential detection of genuine tripartite entanglement. Genuine entanglement is a useful resource for various multiparty communication and information processing tasks. Due to the difficulties present in experimentally producing multipartite quantum correlations, exploring the possibilities of using single multipartite quantum correlation several times is not only interesting for foundational studies but also for several information theoretic applications.

*Chapter 5* :- Chapter 5 discusses the detection of non-Markovianity, another information theoretic resource. In recent times non-Markovian backflow of information is shown to be a useful resource for various information theoretic tasks. Besides Robertson-Schrödinger uncertainty relation, here we use various other uncertainty relations in order to detect the signature of non-Markovianity in an experimentally feasible way.

*Chapter 6* :- Chapter 6 introduces an information theoretic application of another important quantum information theoretic resource called indefinite causal order. By introducing a communication game called '*Random receiver quantum communication*', here we show that the task is only possible with the assistance of *quantum SWITCH* and is not possible by superposing the path degrees of freedom. Thus, the application and importance of indefinite causal order has been established.

*Chapter 7* :- Finally, in chapter 7, we summarize the main results of the thesis with speculations on future directions.

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## MATHEMATICAL AND PHYSICAL PRE-REQUISITES

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In the early twentieth century, black-body radiation, photoelectric effect, Compton scattering and several other astonishing experimental observations were not possible to be interpreted with the then existing theories of classical physics. For the explanation of all these phenomena in a consistent way, a new operational theory called quantum mechanics (QM) was proposed. An operational theory consists of three main postulates– 1) preparation of a state, 2) transformation of the state and 3) measurement of the state. QM, being an operational theory also contains several postulates [40].

### 2.1 POSTULATES OF QUANTUM MECHANICS

#### 2.1.1 Postulate 1 (System and state)

*Any physical system  $S$  is associated to a complex vector space with inner products known as Hilbert space  $\mathcal{H}_S$  and the state of the system is completely described by a density operator  $\rho$ , a positive Hermitian operator on that Hilbert space with  $\text{Tr}(\rho) = 1$ .*

##### 2.1.1.1 Pure state

The state of a quantum system can be categorized in two parts : pure state and mixed state. A state  $\rho_S$  is said to be pure if and only if it is a one dimensional projector acting on  $\mathcal{H}_S$  i.e,  $\rho_S = |\psi\rangle\langle\psi|$  for some  $|\psi\rangle \in \mathcal{H}_S$ . In other words  $\rho_S^2 = \rho_S$ . Thus a pure state can equivalently be specified by a unit vector (wave function),  $|\psi\rangle \in \mathcal{H}_S$ . According to quantum ‘superposition

principle', any linear superposition of pure quantum states is also a valid quantum state. Hence, if  $|\psi_i\rangle \in \mathcal{H}_S, \forall i \in \{1, 2, 3, \dots, n\}$ , then their linear combinations,  $\sum_{i=1}^n c_i |\psi_i\rangle \in \mathcal{H}_S$ . Here,  $c_i (\forall i \in \{1, 2, 3, \dots, n\})$  are complex numbers with  $\sum_{i=1}^n |c_i|^2 = 1$ .

#### 2.1.1.2 Mixed state

A mixed state is nothing but an ensemble or classical statistical mixture of pure states. The necessary and sufficient condition for a state  $\rho_S$  to be a mixed state is that  $\text{Tr}\rho_S^2 < 1$ . Suppose a physical system is in state  $|\psi_i\rangle$  with probability  $p_i$  where  $1 \leq i \leq n$  and  $\sum_i p_i = 1$ , then it is not possible to represent the state of the system with a single vector  $|\psi\rangle \in \mathcal{H}_S$ , rather by a density operator  $\rho_S = \sum_i p_i |\psi_i\rangle\langle\psi_i|$ . Here it is important to mention that a particular mixed state does not have any unique pure decomposition but can be decomposed in infinitely many ways[41, 42].

#### 2.1.1.3 Qubit system

Qubits are two-level quantum systems associated with the two dimensional complex Hilbert space,  $\mathbb{C}^2$  which provide a simple mathematical framework for understanding the basic concepts of QM.

Any state of the qubit system can be represented by a  $2 \times 2$  density operator which is a positive and Hermitian matrix with unit trace. On the other hand, any  $2 \times 2$  Hermitian matrix can be expressed as a linear combination of  $\mathbb{I}, \sigma_x, \sigma_y, \sigma_z$ , where  $\mathbb{I}$  is the identity matrix of dimension two and  $\sigma_i$  (with  $i = x, y, z$ ) are the Pauli matrices. Therefore, in the considered basis  $(\mathbb{I}, \sigma_x, \sigma_y, \sigma_z)$ , the most general form of the state of a qubit system can be written as,

$$\rho = \frac{1}{2} (\mathbb{I} + \vec{n} \cdot \vec{\sigma}), \quad (2.1)$$

where,  $\vec{n} = (n_x, n_y, n_z) \in \mathbb{R}^3$  (three dimensional Euclidean space) and  $\vec{n} \cdot \vec{\sigma} = n_x \sigma_x + n_y \sigma_y + n_z \sigma_z$ .

For pure qubit states, we have  $\text{Tr}(\rho^2) = 1$ , i. e.,  $|\vec{n}| = 1$ . On the other hand, for mixed qubit states, we have  $\text{Tr}(\rho^2) < 1$ , which implies  $|\vec{n}| < 1$ . The state of an arbitrary qubit system can be represented by a point in sphere with unit radius ( $|\vec{n}| \leq 1$ ) known as "Bloch sphere". The

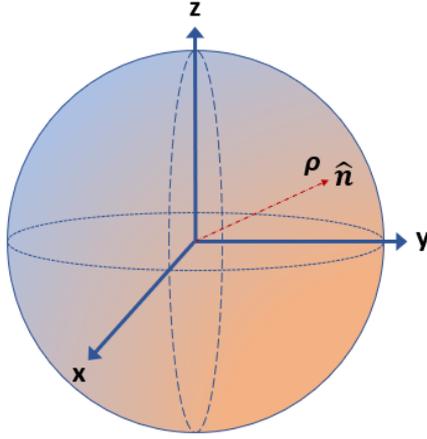


Figure 2.1: Bloch sphere representation for qubit.

points on the surface of the Bloch sphere correspond to the pure qubit states, whereas the interior points correspond to the mixed qubit states.

### 2.1.2 Postulate 2 (Evolution or dynamics)

The time evolution of a closed system is described by a unitary transformation. Suppose the state of a physical system at time  $t_0$  is  $\rho_S(t_0)$  and evolves to some state  $\rho_S(t)$  at a later time  $t$ . Then the time evolution of the state can be written as,

$$\rho_S(t) = U(t_0, t)\rho_S(t_0)U^\dagger(t_0, t) \quad (2.2)$$

Where  $U(t_0, t)$  is the unitary operator depends on the initial time  $t_0$  and final time  $t$ .

For any pure state  $\psi$ , the dynamics is governed by Schrödinger equation,

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle, \quad (2.3)$$

where  $H$  is the Hamiltonian of the system. Thus the unitary evolution can also be written as

$$U(t_0, t) = \exp\left[\frac{-iH(t-t_0)}{\hbar}\right].$$

### 2.1.3 Postulate 3 (Measurement)

Measurements are described by a collection of Hermitian operators  $\{M_m\}$ . These measurement operators  $M_m$  act on the state space  $\mathcal{H}_S$  of the system being measured. The index  $m$  denotes the measurement outcomes that may occur in the experiment. If the state of the quantum system is  $\rho_i$  immediately before the measurement, then the probability of occurring the outcome  $m$  is given by the Born rule,  $p(m) = \text{Tr}(M_m^\dagger M_m \rho_i)$ . The state  $\rho_f^m$  of the system after the measurement, when the outcome  $m$  occurs, is given by,  $\rho_f^m = \frac{M_m \rho_i M_m^\dagger}{\text{Tr}(M_m^\dagger M_m \rho_i)}$ . The measurement operators satisfy the completeness relation,  $\sum_m M_m^\dagger M_m = \mathbb{I}$  where  $\mathbb{I}$  refers to the identity operator on  $\mathcal{H}_S$ .

#### 2.1.3.1 Projective measurement

Projective measurement,  $P$  (von Neumann measurement) is a special class of quantum measurement consisting of a set of mutually orthogonal projectors satisfying the completeness relation,  $P \equiv \{P_m | P_m P_{m'} = \delta_{m,m'} P_m, \sum_m P_m = \mathbb{I}\}$ .

A projective measurement can be described by an observable  $M$ , having spectral decomposition  $M = \sum_m m P_m$ , where  $P_m$  is the projector onto the eigenspace of  $M$  associated with the eigenvalue  $m$  and  $P_m P_{m'} = \delta_{m,m'} P_m$ .

The performance of a projective measurement  $\{P_m\}$  on a state  $\rho$  gives the probability of obtaining the outcome  $m$ ,  $p(m) = \text{Tr}(P_m \rho)$ . The post measurement state  $\rho_f^m$ , when the outcome  $m$  occurs, is given by,  $\rho_f^m = \frac{P_m \rho P_m^\dagger}{\text{Tr}(P_m \rho)}$ . The average value of the observable  $M$  for the state  $\rho$  is given by,

$$\langle M \rangle_\rho = \sum_m m p(m) = \text{Tr}(\rho M). \quad (2.4)$$

#### 2.1.3.2 POVM measurement

A POVM (positive operator valued measurement), as its name suggest is a set of positive operators that add to identity, i.e.,  $E \equiv \{E_i | \sum E_i = \mathbb{I}, 0 < E_i \leq \mathbb{I}\}$ . Projective measurement is a particular class of POVM measurement. When the POVM measurement  $E$  is performed on the state  $\rho$ , the probability of getting  $i$ -th outcome is  $\text{Tr}(\rho E_i)$  where,  $E_i$  are POVM elements. Corresponding to a particular POVM measurement  $E$ , the measurement operators are not

unique. Rather, there exists a set of measurement operators  $\{M_i\}$  defining a measurement corresponding to the POVM  $E \equiv \{E_i\}$ . The general correspondence relation is given as,  $M_i = U_i\sqrt{E_i}$  with  $U_i$  being a unitary operator. Hence, a POVM measurement can be realized in different ways. Therefore, unless the measurement operators corresponding to a POVM measurement are explicitly known, there remains an ambiguity about the post measurement state after the POVM measurement.

#### 2.1.4 Postulate 4 (Composite system)

The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. If the state space of the  $i^{\text{th}}$  system ( $i \in \{1, 2, \dots, n\}$ ) is  $\mathcal{H}_{S_i}$ , then the state space of a composite physical system is  $\mathcal{H}_{S_1} \otimes \mathcal{H}_{S_2} \otimes \mathcal{H}_{S_3} \otimes \dots \otimes \mathcal{H}_{S_n}$ . Moreover, if the  $i^{\text{th}}$  system is prepared in the state  $\rho_i$ , then the joint state of the composite system is  $\rho_1 \otimes \rho_2 \otimes \rho_3 \otimes \dots \otimes \rho_n$ .

Let us consider a simplest example of such composite systems. For this, consider a two-qubit system. The quantum state of two qubits (operator on  $\mathbb{C}^2 \otimes \mathbb{C}^2$ ) in the Hilbert-Schmidt basis takes the form,

$$\rho_{1,2} = \frac{1}{4}(\mathbb{I} \otimes \mathbb{I} + \vec{r} \cdot \vec{\sigma} \otimes \mathbb{I} + \mathbb{I} \otimes \vec{s} \cdot \vec{\sigma} + \sum_{i,j=1}^3 t_{ij}\sigma_i \otimes \sigma_j). \quad (2.5)$$

Here,  $\mathbb{I}$  is identity operator acting on  $\mathbb{C}^2$ ,  $\sigma_i$  represent Pauli matrices and  $\vec{r}, \vec{s} \in \mathbb{R}^3$ .  $\vec{r} \cdot \vec{\sigma} = \sum_{i=1}^3 r_i \sigma_i$ .  $\vec{s} \cdot \vec{\sigma} = \sum_{i=1}^3 s_i \sigma_i$ . The coefficients  $t_{ij} = \text{Tr}(\rho \sigma_i \otimes \sigma_j)$  form the correlation matrix  $T$ . In addition, since  $\rho$  is a valid density matrix,  $\rho \geq 0$  and  $\text{Tr}(\rho) = 1$ .

Given the state of a composite system, one may be interested to find the state of any subsystems. The physical state of the subsystem can be obtained from the state of the composite system by performing partial trace operation. For example, given a state,  $\rho_{12}$  of the composite system which is made up of two subsystems 1 and 2, the reduced state of the subsystem 1 can be calculated as  $\rho_1 = \text{Tr}_2 \rho_{12}$ . Here,  $\text{Tr}_2$  is the partial trace over the subsystem 2.

## 2.2 BELL NONLOCALITY

In this section we discuss about Bell nonlocality which was discovered by J. S. Bell by following upon the EPR paradox. In 1935, taking the assumption of local-realism EPR showed that QM is not a complete theory [4]. Much later, in 1965, Bell [7] provided an empirically testable criteria called Bell inequality [7, 8, 43] to check whether correlations of an operational theory (not only QM) can be explained in a local-realistic framework. Using Bell inequality, it was shown that there exists certain quantum correlations which are not compatible with the assumptions of local-realism. Bell inequality thus pointed out the difference between quantum and the classical world. Here, we elaborate Bohm's version [44] of EPR paradox and Bell inequality.

### 2.2.1 EPR argument

Quantum mechanics is an operational theory and is fundamentally probabilistic in nature. But this probability is not due to any subjective ignorance about the pre-assigned value of a dynamical variable in a quantum state, rather it represents the probability of finding a particular value of a dynamical variable if that dynamical variable is measured. But if the measurements are not performed, QM cannot tell anything about the value of a dynamical variable of a system. Therefore, physicists including Einstein were not at all satisfied with this probabilistic interpretation of quantum mechanics.

In 1935, Einstein along with his two colleagues, Podolsky and Rosen (EPR) came up with their seminal paper [4] to establish that QM as an operational theory though consistent, is not complete in its present form. They approached by introducing the following assumptions:

- **Necessary condition for completeness:** "Every element of physical reality must have a counterpart in that physical theory".
- **Sufficient condition for reality:** "If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity), the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity."

- **Locality principle:** “Elements of reality pertaining to one system can not be affected instantaneously by measurements performed at a distance on another system.”

Initially, EPR showed the incompatibility of the above assumptions considering a position-momentum correlated state and concluded that QM is incomplete. It was D. Bohm [44], who later presented a simpler form of the EPR argument using spin- $\frac{1}{2}$  particles. Let us now discuss Bohm’s version of the original EPR argument.

Consider two spatially separated observers say, Alice and Bob. They perform measurements on two spin- $\frac{1}{2}$  particles which used to interact in the past. The joint state in the  $z$ -basis (eigenstate of  $\sigma_z$ ) can be represented as

$$|\psi_{\text{singlet}}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B). \quad (2.6)$$

Here,  $|0\rangle, |1\rangle$  are the two eigenstates of  $\sigma_z$  operator with eigenvalues  $+1$  and  $-1$  respectively and subscript  $A$  and  $B$  represents the particle of Alice and Bob respectively. Above state has a nice property that it is invariant under all spatial rotations and can be written in various equivalent ways. For instance, the state can be rewritten in the  $x$ -basis (eigenvector of  $\sigma_x$ ) as

$$|\psi_{\text{singlet}}\rangle = \frac{1}{\sqrt{2}}(|+\rangle_A|-\rangle_B - |-\rangle_A|+\rangle_B), \quad (2.7)$$

where  $|+\rangle, |-\rangle$  are the two eigenstates of  $\sigma_x$  operator with eigenvalues  $+1$  and  $-1$  respectively.

Now, if Alice performs measurement on her spin- $\frac{1}{2}$  particle along the  $z$ -direction (observables associated to the operator  $\sigma_z$ ) the outcome will be either  $+1$  or  $-1$  and, correspondingly, the state of the particle will collapse on  $|0\rangle_A$  or  $|1\rangle_A$  respectively. But, because of the correlation present in the singlet state the state of the Bob’s particle will also collapse on either  $|1\rangle_B$  or  $|0\rangle_B$  instantaneously. Alice can, therefore, predict the value of the spin along  $z$ -direction of Bob’s particle without any way disturbing it (which is spatially separated from Alice’s system). Now, invoking the locality principle one can say that for Bob’s particle, the value of the spin along  $z$ -direction had a definite value even before Alice performed the measurement on particle  $A$ , as distant measurement can not create some physical reality for another system. Thus, according to EPR, there exists an element of physical reality connected with the  $\sigma_z$

measurement. Similarly, Alice could measure  $\sigma_x$  and predict with certainty, without disturbing the system, the outcome of a possible  $\sigma_x$  measurement by Bob. Hence, there also exists an element of reality corresponding to the  $\sigma_x$  measurement. Therefore, it can be claimed that Bob's particle possesses element of physical reality for both the observables  $\sigma_z$  and  $\sigma_x$ . However, for spin- $\frac{1}{2}$  system, the operators  $\sigma_z$  and  $\sigma_x$  do not commute and, hence, there is no common eigenstate for both the observables. Hence, QM can not assign reality to both the observables. In other words, although above situation leads to the existence of simultaneous reality for both the observables associated with the operators  $\sigma_z$  and  $\sigma_x$  but in any situation, in the framework of QM, it is not possible to assign reality to these two observables simultaneously. Thus, QM is an incomplete theory.

### 2.2.2 *Local realism and Bell's theorem*

After the argument of EPR, it is natural to ask whether QM can be completed by supplementing some unknown, extra variables, known as 'hidden variables' to the quantum state. Any possibilities of supplementing such variables to the quantum state will in turn give a complete description of the quantum state in the sense that there are some well defined and pre-existing values for every variable, and measurements just reveal those pre-existing values. But no one thought in this line since von Neumann [45] long ago discarded this approach. Surprisingly, it was J. S. Bell who again posed this question whether any local realistic theory can reproduce all the statistical results of quantum mechanics in operational level though there may exist some complete description of state in the ontological level [7]. With this profound approach Bell derived certain constraints in the form of inequality known as Bell inequality. He gave an experimentally testable criteria and showed that any physically observed correlations coming from some local realistic theory must satisfy this inequality and any violation of this inequality will demonstrate the non-existence of local realistic description of that theory.

In order to understand the essence of Bell inequality [7, 8, 43], let us consider a joint system consisting of two subsystems shared between two spatially separated parties say Alice and Bob. The preparation of the joint system is denoted by  $\rho$ . Both of the observers have access to

their respective subsystem where they can perform any local measurements. Let us consider that Alice has two measurements  $A_0$  and  $A_1$  which she can perform randomly on her subsystem. Similarly, Bob also has two measurements,  $B_0$  or  $B_1$  which he can choose to perform independently and randomly on his subsystem. Let the corresponding measurement results be  $a_0, a_1, b_0, b_1 \in \{+1, -1\}$ . Repeating the above measurements a number of times on their respective part they get the correlation  $P(a_i, b_j|A_i, B_j, \rho) = \{p(a_i, b_j|A_i, B_j, \rho)\}_{a_i, b_j, A_i, B_j}$  which is the set of joint probability distributions  $p(a_i, b_j|A_i, B_j, \rho) \forall a_i, b_j, A_i, B_j$ . Here,  $p(a_i, b_j|A_i, B_j, \rho)$  represents the joint probability of getting outcome  $a_i$  and  $b_j$  when Alice performs the measurements  $A_i$  and Bob performs the measurements  $B_j$  respectively on the joint system  $\rho$ . For any pair of measurements,  $A_i$  and  $B_j$ , the joint probability can be calculated as,

$$\langle A_i, B_j \rangle = \sum_{a_i, b_j = -1}^{+1} a_i b_j p(a_i, b_j|A_i, B_j, \rho). \quad (2.8)$$

Let  $\lambda$  be a local-realistic complete state associated with this joint system. The conditional joint probability in the hidden variable theory is denoted by  $p(a_i, b_j|A_i, B_j, \rho, \lambda)$ . This hidden variable theory is however required to reproduce the observed joint probability distributions in the operational theory,

$$p(a_i, b_j|A_i, B_j, \rho) = \int \mu(\lambda) p(a_i, b_j|A_i, B_j, \rho, \lambda) d\lambda \quad \forall a_i, b_j, A_i, B_j. \quad (2.9)$$

Let us now consider the assumptions under which Bell inequality was derived.

**Realism (or Determinism) :** A hidden variable theory is said to satisfy realism (also called determinism) if and only if

$$p(a_i, b_j|A_i, B_j, \rho, \lambda) \in \{0, 1\} \quad \forall a_i, b_j, A_i, B_j. \quad (2.10)$$

**Locality :** A hidden variable theory is said to satisfy locality if and only if

$$\begin{aligned} p(a_i|A_i, B_j, \rho, \lambda) &= p(a_i|A_i, \rho, \lambda) \quad \forall a_i, A_i, B_j \\ p(b_j|A_i, B_j, \rho, \lambda) &= p(b_j|B_j, \rho, \lambda) \quad \forall b_j, A_i, B_j. \end{aligned} \quad (2.11)$$

Let us now show that if a hidden variable theory satisfies the assumptions of reality and locality, then the joint probability distribution  $p(a_i|A_i, B_j, \rho, \lambda)$  becomes factorisable as the product of  $p(a_i|A_i, \rho, \lambda)$  and  $p(b_j|B_j, \rho, \lambda) \forall a_i, b_j, A_i, B_j$ .

$$\begin{aligned}
p(a_i|A_i, B_j, \rho, \lambda) &= p(a_i|A_i, B_j, \rho, \lambda, b)p(b_j|A_i, B_j, \rho, \lambda) \quad (\text{Bays' rule}) \\
&= p(a_i|A_i, B_j, \rho, \lambda)p(b_j|A_i, B_j, \rho, \lambda) \quad (\text{Determinism condition}) \\
&= p(a_i|A_i, \rho, \lambda)p(b_j|B_j, \rho, \lambda) \quad (\text{Locality condition}). \quad (2.12)
\end{aligned}$$

Thus, the conjunction of the reality and locality assumption, leads to the following ‘‘factorisability condition’’ of the correlation;

$$p(a_i|A_i, B_j, \rho, \lambda) = \int \mu(\lambda)p(a_i|A_i, \rho, \lambda)p(b_j|B_j, \rho, \lambda)d\lambda. \quad (2.13)$$

Under above condition, the expectation value  $\langle A_i B_j \rangle$  can be expressed as

$$\begin{aligned}
\langle A_i B_j \rangle_\rho &= \sum_{a_i, b_j = -1}^{+1} a_i b_j \int \mu(\lambda)p(a_i|A_i, \rho, \lambda)p(b_j|B_j, \rho, \lambda)d\lambda \\
&= \int d\lambda \mu(\lambda) \sum_{a_i = -1}^{+1} a_i p(a_i|A_i, \rho, \lambda) \sum_{b_j = -1}^{+1} b_j p(b_j|B_j, \rho, \lambda) \\
&= \int d\lambda \mu(\lambda) \langle A_i \rangle_\rho^\lambda \langle B_j \rangle_\rho^\lambda, \quad (2.14)
\end{aligned}$$

where  $\langle A_i \rangle_\rho^\lambda = \sum_{a_i = -1}^{+1} a_i p(a_i|A_i, \rho, \lambda) \in \{0, 1\}$  and  $\langle B_j \rangle_\rho^\lambda = \sum_{b_j = -1}^{+1} b_j p(b_j|B_j, \rho, \lambda) \in \{0, 1\}$ .

Let us now consider the quantity  $B_{CHSH}$  defined as:

$$|\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle|. \quad (2.15)$$

This is a quantity of the operation level and can be obtained from  $B_{CHSH}^\lambda$  by integrating over the  $\lambda$  space, i.e,  $B_{CHSH} \equiv \int \mu(\lambda) |B_{CHSH}^\lambda| d\lambda$ , where  $B_{CHSH}^\lambda$  can be defined as,

$$\begin{aligned}
B_{CHSH}^\lambda &= \langle A_0 \rangle^\lambda \langle B_0 \rangle^\lambda + \langle A_0 \rangle^\lambda \langle B_1 \rangle^\lambda + \langle A_1 \rangle^\lambda \langle B_0 \rangle^\lambda - \langle A_1 \rangle^\lambda \langle B_1 \rangle^\lambda \\
&= \langle A_0 \rangle^\lambda (\langle B_0 \rangle^\lambda + \langle B_1 \rangle^\lambda) + \langle A_1 \rangle^\lambda (\langle B_0 \rangle^\lambda - \langle B_1 \rangle^\lambda). \quad (2.16)
\end{aligned}$$

Now since  $\langle A_i \rangle^\lambda \in \{0, 1\}$  and  $\langle B_j \rangle^\lambda \in \{0, 1\} \quad \forall \{i, j\}$ , it is straightforward to see that for any fixed  $\lambda$

$$B_{CHSH}^\lambda = \pm 2. \quad (2.17)$$

Now, the average of  $B_{CHSH}$  over some distribution  $\mu(\lambda)$  of hidden variables  $\lambda$  is

$$-2 \leq \langle B_{CHSH} \rangle = \int \mu(\lambda) B_{CHSH}(\lambda) d\lambda \leq +2. \quad (2.18)$$

Thus we obtain the following famous Bell-CHSH inequality in terms of experimentally observable correlation functions  $\langle A_i B_j \rangle, i, j \in \{0, 1\}$

$$|\langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle| \leq 2. \quad (2.19)$$

The conjunction of two assumptions, reality and locality, in the hidden variable theory implies satisfaction of Bell inequality 2.19 by the corresponding operational theory [7, 8, 43]. However, the astonishing fact is that those constraints are not always compatible with the predictions of quantum mechanics. There is some entangled or non-separable state, for which the correlations cannot be always explained through some local-realistic model and are also verified experimentally [9, 10].

### 2.2.3 Quantum violation of Bell inequality

According to Bell's theorem, if QM can be replaced by some hypothetical local hidden variable model then the expression of Bell inequality will be bounded by 2. However, there exist some states and for suitably chosen measurements, Bell inequality is shown to be violated by QM. Let us consider a singlet state  $|\psi_{\text{singlet}}\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B)$  shared between two distant and spatially separated parties say Alice and Bob. Alice and Bob can perform local quantum measurements on their respective parts. Suppose, the measurement operators on

Alice's side are given by,  $A_0 = \frac{\sigma_z + \sigma_x}{\sqrt{2}}$  and  $A_1 = \frac{\sigma_z - \sigma_x}{\sqrt{2}}$  where as the measurement operators on Bob's side are given by,  $B_0 = \sigma_z$  and  $B_1 = \sigma_x$ . With the above mentioned choice of measurement observables and shared singlet state the Bell-CHSH expression can be evaluated as  $B_{CHSH} = 2\sqrt{2}$ . This shows a clear example of violation of Bell-CHSH inequality by QM which implies that even if QM can be completed by a hidden variable theory, that hidden variable theory will violate either realism or locality or both of them.

On the other hand one can see that the algebraic maximum of Bell-CHSH expression  $B_{CHSH}$  is 4. However, for any choice of quantum states and measurements, QM mechanics can not reach the algebraic maximum rather the maximum quantum violation of Bell-CHSH inequality is  $2\sqrt{2}$ , which is known as Cirel'son bound [46].

### 2.3 EPR STEERING

After the argument of Einstein, Podolsky and Rosen [4] where they conjectured that nonlocality is an artefact of the incompleteness of the quantum mechanics, Schrödinger [5, 6] coined the term *steering*, to describe how the choice of a measurement basis on one side could affect the state on the other. He however, believed that steering would never be observed experimentally, since there existed on the other side a definite state (Local Hidden State) even if it is unknown to the local observer. Demonstration of EPR-steering, on the other hand, was first proposed by Reid [47], with subsequent experimental realization by Ou *et al.* [48] and others [49]. Recently, Wiseman *et al.* [11] formulated the operational definition of EPR steering in terms of an information theoretic tasks. The task is as follows. Alice prepares a bipartite quantum state and sends one part to Bob. She can repeat as many times as she wants. Each time they measure their respective parts and communicate classically. Alice's task is to convince Bob that the state she shared is entangled. Bob on the other hand believes that the measurements he performed obey the laws of QM. He, however, does not trust Alice. If the correlations obtained between Alice and Bob's measurement result can be explained by a LHS model in Bob's side then Bob will not be convinced that the state is entangled. Alice could have drawn a pure state at random from some ensemble and sent it to Bob, and then chosen her result based on her knowledge of this LHS. Conversely, if the correlations cannot

be so explained then the state must be entangled. Therefore, Alice will succeed in her task iff she can create genuinely different ensembles for Bob by steering Bob's state. Note that in EPR steering scenario the no-signalling condition is always satisfied by the two parties. Unlike entanglement and nonlocality, EPR steering is inherently asymmetric with respect to the two observers [50].

Let us discuss the steering scenario more elaborately. Alice and Bob are spatially separated and an unknown quantum state say,  $\rho_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$  is shared among them. Here  $\mathcal{H}_A$  and  $\mathcal{H}_B$  are the Hilbert space of Alice's and Bob's subsystems respectively. Alice performs some uncharacterized or black-box measurements  $\{M_{a|x}\}_{a,x}$  ( $M_{a|x} \geq 0 \forall a, x$ ; and  $\sum_a M_{a|x} = \mathbb{I} \forall x$ ) on her respective part of the shared bipartite state  $\rho_{AB}$ . Due to the measurements performed on Alice's particle some conditional states are prepared on Bob's side. The set of unnormalized conditional states on Bob's side  $\{\sigma_{a|x}\}_{a,x}$  are called the assemblages. The elements of the assemblage are characterized by the conditional probability of obtaining the outcome  $a$  when Alice performs the measurement  $x$  and the normalized conditional state  $\rho_{a|x}$  on Bob's side i.e,  $\sigma_{a|x} = p(a|x)\rho_{a|x}$ . According to the rule of QM, the following criteria should be satisfied by the assemblages.

$$\sigma_{a|x} = \text{Tr}_A[(M_{a|x} \otimes \mathbb{I})\rho_{AB}] \quad \forall \sigma_{a|x} \in \{\sigma_{a|x}\}_{a,x} \quad (2.20)$$

If the assemblage does not contain any local hidden state (LHS) model on Bob's side, i.e., if  $\forall a, x, \sigma_{a|x}$  cannot be decomposed in the form,

$$\sigma_{a|x} = \sum_{\lambda} p(\lambda) p(a|x, \lambda) \rho_{\lambda}, \quad (2.21)$$

then Alice succeeded in demonstrating steerability to Bob. Here  $\lambda$  corresponds the local hidden variable (LHV) which occurs with probability  $p(\lambda)$  and the quantum states  $\rho_{\lambda}$  denotes local hidden states satisfying  $\rho_{\lambda} \geq 0$  and  $\text{Tr}(\rho_{\lambda}) = 1$ .

Bob, now performs a set of characterized POVM measurements  $\{M_{b|y}\}_{b,y}$  ( $M_{b|y} \geq 0 \forall b, y$ ; and  $\sum_b M_{b|y} = \mathbb{I} \forall y$ ) on the conditional states prepared by Alice's unknown measurements and obtain  $p_Q(b|y, \rho_{\lambda}) = \text{Tr}(M_{b|y}\sigma_{a|x})$ .

Then the joint probability distribution  $P(ab|xy)$  detects the steerability from Alice's side to Bob if and only if it does not have a decomposition as follows:

$$p(ab|xy) = \sum_{\lambda} p(\lambda) p(a|x, \lambda) p_Q(b|y, \rho_{\lambda}) \quad \forall a, x, b, y; \quad (2.22)$$

where,  $\sum_{\lambda} p(\lambda) = 1$ ,  $p(a|x, \lambda)$  denotes an arbitrary probability distribution arising from LHV  $\lambda$  and  $p_Q(b|y, \rho_{\lambda})$  denotes the quantum probability of occurring the outcome  $b$  when the quantum measurement  $y$  is performed on the LHS  $\rho_{\lambda}$ . Hence, the joint probability distribution  $P(ab|xy)$  will be called steerable if and only if it does not have a LHV-LHS model.

#### 2.4 ENTANGLEMENT

Consider a bipartite state  $\rho_{AB}$ , consisting of two subsystems  $A$  and  $B$ . The state is said to be separable if and only if the state can be decomposed in the following form,

$$\rho_{AB} = \sum_i p_i \rho_i^A \otimes \rho_i^B, \quad (2.23)$$

with  $\sum_i p_i = 1$  and  $\rho_i^A$ s and  $\rho_i^B$ s are the state on the subsystem  $A$  and  $B$  respectively. On the other hand, if a bipartite state cannot be written in the above form then the state is called entangled [51].

For the case of pure states, a bipartite state  $|\psi\rangle_{AB}$ , consisting of two subsystems  $A$  and  $B$ , is said to be entangled if and only if the state cannot be written in the following product form,

$$|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B. \quad (2.24)$$

Here  $|\psi\rangle_A$  and  $|\psi\rangle_B$  are the states representing the subsystem  $A$  and  $B$  respectively.

The necessary and sufficient criterion for the detection of entanglement in a qubit-qubit system (in  $\mathbb{C}^2 \otimes \mathbb{C}^2$ ) and qubit-qutrit system (in  $\mathbb{C}^2 \otimes \mathbb{C}^3$ ) is given by the Peres-Horodecki positive partial transposition (PPT) condition. It states that a qubit-qubit state or a qubit-qutrit state is entangled if and only if the partial transposition of that state contains at least

one negative eigenvalue. However, this Peres-Horodecki PPT criterion is not sufficient to detect entanglement for higher dimensional bipartite system (e.g, system having dimension other than  $\mathbb{C}^2 \otimes \mathbb{C}^2$  or  $\mathbb{C}^2 \otimes \mathbb{C}^3$ ) [51, 52].

Note that Bell nonlocality, EPR steering, entanglement are equivalent for all pure states, i.e., any pure entangled state is EPR steerable as well as Bell nonlocal. However, for the case of mixed entangled states, the states showing Bell nonlocality form a strict subset of mixed states showing EPR steering, which again form a strict subset of mixed entangled states.

#### 2.4.1 Genuine entanglement

A tripartite state  $\rho$  is called bi-separable if and only if it can be decomposed in the following form,

$$\rho = \sum_{\lambda} p_{\lambda} \rho_{\lambda}^A \otimes \rho_{\lambda}^{BC} + \sum_{\mu} p_{\mu} \rho_{\mu}^B \otimes \rho_{\mu}^{AC} + \sum_{\nu} p_{\nu} \rho_{\nu}^C \otimes \rho_{\nu}^{AB}, \quad (2.25)$$

with  $0 \leq p_{\lambda}, p_{\mu}, p_{\nu} \leq 1$  and  $\sum_{\lambda} p_{\lambda} + \sum_{\mu} p_{\mu} + \sum_{\nu} p_{\nu} = 1$ . A tripartite state is said to be genuinely entangled if and only if it cannot be written in the bi-separable form (2.25) [53].

## 2.5 OPEN QUANTUM SYSTEM

An open quantum system deals with the quantum system interacting with its surroundings known as environment or bath. Due to the interaction with noisy environment, the dynamics of the system changes significantly enabling physical explanations for dissipation, decoherence and other irreversible phenomena[23, 24, 54-58].

Let  $\rho(t) \in S$  symbolize the physical state of a closed system at time  $t$  with  $S$  denoting the set of density matrices acting on some Hilbert space  $\mathcal{H}_S$ . The time evolution of the state of the system is governed by von Neumann equation

$$\frac{d}{dt} \rho(t) = -i [H(t), \rho(t)]. \quad (2.26)$$

Here,  $[A, B] \equiv AB - BA$  represents the commutator of  $A$  and  $B$  and  $H(t)$  is the time dependent Hamiltonian of the system. The solution of this equation is the unitary evolution:

$$\rho(t) = U(t)\rho U^\dagger(t), \quad U(t) = \mathcal{T} \exp \left( -i \int_0^t ds H(s) \right). \quad (2.27)$$

Instead of a closed quantum system, if an open quantum system is considered then the evolution of the system alone cannot be represented by unitary operation since there is an interaction with an additional degree of freedom called the bath or environment. The overall evolution of an open quantum system is always a physical process and maps a set of density operators to itself.

Let  $\rho_S(0)$  be the initial density operator (i.e, at  $t = 0$ ) of an open quantum system. Then, the density operator  $\rho_S(t)$  at some later time  $t$  can be evaluated as  $\Lambda(t)(\rho_S(0)) = \rho_S(t)$ , where  $\Lambda(t)$  is the evolution map. This evolution map  $\Lambda(t)$  maps the set of density operator  $S(\mathcal{H}_S)$  (which act on the Hilbert space  $\mathcal{H}_S$ ) to itself, i.e,  $\Lambda(t) : S(\mathcal{H}_S) \rightarrow S(\mathcal{H}_S)$ . This kind of map represents the state of the system in different times and are also known as dynamical map. Let  $H_S$  be the Hamiltonian of the system under consideration,  $H_B$  be the Hamiltonian associated to the bath degrees of freedom and  $H_I$  be the interaction Hamiltonian between system and bath. The overall evolution of the system and the bath is depicted by the unitary operator,

$$U(t) = \exp(-iH(t)),$$

where  $H = H_S + H_B + H_I$  is the total Hamiltonian of the system. In the simplest scenario, consider that initially, the system and the bath are uncorrelated. Hence the joint state of the system and the bath can be written as,

$$\rho(0) = \rho_S(0) \otimes \rho_B. \quad (2.28)$$

After some time  $t$ , the state evolves to

$$\rho(t) = U(t)(\rho_S(0) \otimes \rho_B)U^\dagger(t). \quad (2.29)$$

Now, the state of the system alone can be evaluated by tracing out the bath degrees of freedom from the joint state  $\rho(t)$ ,

$$\rho_S(t) = \text{Tr}_B(U(t)(\rho_S(0) \otimes \rho_B)U^\dagger(t)). \quad (2.30)$$

Hence,

$$\rho_S(t) = \Lambda(t)(\rho_S(0)) = \text{Tr}_B(U(t)(\rho_S(0) \otimes \rho_B)U^\dagger(t)). \quad (2.31)$$

Let us now draw a clearer picture of the dynamical map  $\Lambda(t)$ . Any map  $\Lambda(t)$  preserves the following properties,

1) **Hermiticity** :  $\Lambda(t)(A^\dagger) = \Lambda(t)(A)^\dagger$ , which implies that the dynamical map  $\Lambda(t)$  preserves the hermiticity.

2) **Positivity** : The map  $\Lambda(t)$  is said to be positive if and only if it preserves the positivity of an operator i.e, it maps a positive operator to another positive operator. Mathematically,

$$\Lambda(t)(\rho) \geq 0, \forall \rho \geq 0 \text{ where } \rho \in S.$$

3) **Complete Positivity** : The map is called completely positive if and only if the composite map  $\mathbb{I}_k \otimes \Lambda(t)$  is also positive for all  $k$ . Here  $\mathbb{I}_k$  is the identity map on a  $k$ -dimensional density operators which is greater than or equal to the dimension of the space on which  $\Lambda(t)$  acts.

4) **Trace condition** : The map is said to be trace preserving if and only if  $\text{Tr}[\Lambda(t)(\rho)] = \text{Tr}[\rho]$  for all positive operators  $\rho$ .

Analogous to a closed system where unitary operations are the most general operations that map a quantum state to another quantum state, a completely positive map is the most general mathematical entity which maps a density operator to another density operator. The complete-positivity and trace preserving properties of the dynamical map  $\Lambda$  ensures that the output state of the map will also be a valid density matrix. These kind of dynamical maps are also known as completely positive trace preserving maps or CPTP-maps.

There are numerous representations for completely-positive maps e.g, Stinespring representation, the Kraus or the operator-sum representation, the Choi-matrix representation etc.

**The operator-sum or Kraus representation** : A CPTP map,  $\Lambda$  always allows a Kraus decomposition of the form  $\Lambda(\rho) = \sum_i K_i \rho K_i^\dagger$ , with  $K_i \in \mathcal{B}(\mathcal{H})$  are called Kraus operators and

$\sum_i K_i^\dagger K_i = \mathbb{I}$ . Here,  $\mathbb{I}$  represents the identity matrix for arbitrary dimensions. For a given map  $\Lambda$ , the set of Kraus operators,  $\{K_i\}$  are not unique and all such sets of Kraus operators, representing the same map  $\Lambda$  can be connected by a unitary transformation.

**The Choi-matrix representation :** Consider an arbitrary quantum dynamical map,  $\Lambda(t, t_0) : \rho(t_0) \rightarrow \rho(t)$ . Let us denote a set  $\mathcal{D}$  which contains all such quantum channels. According to the channel-state duality [59, 60], one can always find a one to one connection of the set  $\mathcal{D}$  to the set of all Choi-states  $\mathcal{F}$ , where the Choi-state corresponding to any quantum evolution  $\Lambda(t, t_0)$  can be represented as

$$\mathcal{C}_\Lambda(t, t_0) = \mathbb{I} \otimes \Lambda(t, t_0)(|\phi\rangle\langle\phi|).$$

Here  $|\phi\rangle\langle\phi|$  is a  $d \times d$  dimensional maximally entangled state for a  $d$  dimensional system.

As mentioned earlier, for an open quantum system, due to the interaction of the system with the environment, the dynamics of the system cannot be described by the unitary operation alone. Hence, it is not easy to solve the dynamics of an open quantum system. However, one may consider system-bath compositely which together evolve in a unitary fashion. In such a scenario, one may determine the dynamics of the system alone from that of the combined (system+bath) evolution by averaging over the bath degrees of freedom. Such dynamical evolution of the system is represented by master equation. The simplest quantum master equation is the Markovian master equation which describes the evolution of the system assuming it's interaction with a memoryless environment. More informally, a Markovian master equation can also be described in such a way that any information transfer from the system to the environment is actually a one way traffic and has no influence on the system dynamics at later times. This is called 'Markov-approximation'. There is also another well known approximation called 'Born-approximation' which states that when the interaction between the system and it's environment is sufficiently weak, and when the bath is large enough compared to the system then any changes to the combined system over time can be approximated as the changes on the system only and the bath is almost unaltered. Putting in several such subsequent approximations, the evolution of the system can thus be easily derived. There are several other forms of Master equation but the simplest Markovian mas-

ter equation is called the Lindblad equation which is a generalization of the von Neumann equation with an additional dissipator term. The master equation is of the form

$$\frac{d}{dt}\rho(t) = -i[H(t), \rho(t)] + \sum_j \gamma_j D[A_j]\rho(t), \quad (2.32)$$

with  $D[A_j]$  being the Lindblad dissipator which generates the dissipative evolution at a rate  $\gamma_j$ . A detailed form of the Lindblad dissipator can be represented as,

$$D[A]\rho(t) = A\rho(t)A^\dagger - \frac{1}{2} \{A^\dagger A, \rho(t)\}. \quad (2.33)$$

Here, the term  $\{A, B\} \equiv AB + BA$  is called the anti-commutator of  $A$  and  $B$ . This type of dynamics are in general not reversible and hence are responsible for the decoherence mechanisms in a system. The solution of such a Lindblad master equation is a CP-map. These CP-maps that represents the dynamics of an open quantum system are also referred to as quantum channels.

Usually, the general quantum evolution can be classified into two specific classes namely Markovian and non-Markovian. Markovian evolutions can be recognized by those special types of evolutions, for which the information flow from the system to the environment is a one way traffic. These types of evolutions can be represented by divisible quantum maps [24, 57, 58]. However, there exist situation when the system-environment coupling is not sufficiently weak or the environment is non-stationary, the Born-Markov approximation may not hold. For such realistic scenario, indivisible dynamics can also occurs which in turns can provide the signatures of non-Markovian information backflow [25, 26, 61–64].

**Divisibility :** Consider that the inverse of  $\Lambda(t)$  exists for all times  $t \geq 0$ . Then a two-parameter family of maps can be defined as

$$\Lambda(t, s) = \Lambda(t)\Lambda^{-1}(s), \quad 0 \leq s \leq t.$$

Here,  $\Lambda(t, 0) = \Lambda(t)$  and  $\Lambda(t, 0) = \Lambda(t, s)\Lambda(s, 0)$ . The existence of the inverse for all times thus guarantees the divisibility of the dynamics. While  $\Lambda_{t,0}$  and  $\Lambda_{s,0}$  are completely positive by construction, the map  $\Lambda(t, s)$  need not be completely positive and not even positive since

the inverse,  $\Lambda^{-1}(s)$  of a completely positive map,  $\Lambda(s)$  need not be positive. The family of dynamical maps are called to be P divisible if  $\Lambda(t, s)$  is positive, and CP divisible if  $\Lambda(t, s)$  is completely positive for all  $0 \leq s \leq t$ .

Now, the Choi-Jamiolkowski (CJ) isomorphism [59, 60] implies that for complete-positivity (CP) of the dynamics  $\Lambda(t, t_0)$ , it is sufficient to show the corresponding Choi state  $\mathcal{C}_\Lambda(t, t_0) \geq 0, \forall(t, t_0)$ .

The channel or dynamical map is called CP-divisible iff it can always be written in the form  $\Lambda(t, t_0) = \Lambda(t, t_1) \circ \Lambda(t_1, t_0)$  where  $t > t_1 > t_0 \forall t, t_1, t_0$ . This is equivalent to the condition that the dynamics is CP for all time intervals [25].

## 2.6 INDEFINITE CAUSAL ORDER

So far we see several quantum features leading to various counter intuitive phenomena which otherwise are not possible in its classical counterpart. Most of these phenomena exploit superposition in space degrees of freedom. However, causal relation between events remains fixed in quantum theory. Nevertheless, in general theory of relativity the space-time itself is dynamical and local clocks are affected by the presence of massive objects. Hence, the causal relation between the events are defined with respect to the presence of massive objects. However, the dynamical causal structure is still classically predefined in general theory of relativity and the causal relation between any pair of events are uniquely determined by the distribution of matter-energy in the past light-cone. But, behind this well-defined causal structure the mass-energy distribution is taken as classical variable. Surprisingly, if quantum states of gravitating degrees of freedom are considered i.e, if a massive system is composed in such a way that it is in a superposition of two distinct states, then the causal relations between events may manifest genuine quantum features. In recent times, there is significant research interest to investigate physically relevant situations where the causal ordering between events is itself governed by quantum mechanics and hence may become indefinite [30, 31].

Recently, application of indefinite causal orders have attracted much interest. Among various information processing tasks, the quantum advantage of indefinite causal order is es-

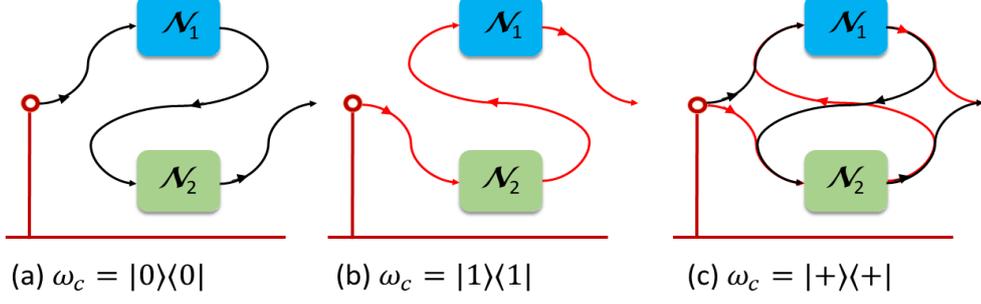


Figure 2.2: Schematic representation of *Quantum SWITCH*.

established in testing properties of quantum channels [36], winning nonlocal games [30, 31], reducing quantum communication complexity [37], enhancing classical and quantum information capacity of noisy quantum channels [38, 39] and many other things.

### 2.6.1 *Quantum SWITCH*

A canonical example of indefinite causal order results from *quantum SWITCH* where relative order of two channels is made entangled with a control system [36]. In a series of recent works, quantum superposition of causal order especially the *quantum SWITCH* has been identified as useful resource in communication tasks as it can offer enhancement of classical as well as quantum channel capacity of noisy quantum channels [34, 38, 39, 65].

A quantum channel is in general represented by a completely positive trace preserving (CPTP) map. The action of the quantum channel  $\mathcal{N}$  on a state  $\rho$  can be represented by Kraus decomposition;  $\mathcal{N}(\rho) = \sum_i K_i \rho K_i^\dagger$ , where  $\{K_i\}$  is a set of Hermitian operators such that  $K_i^\dagger K_i = \mathbb{I}$ .

Let us consider two channels  $\mathcal{N}_1$  and  $\mathcal{N}_2$  having Kraus representation  $\{K_i^{(1)}\}$  and  $\{K_i^{(2)}\}$  respectively. In a fixed causal structure, two channels can be arranged either in series or in parallel. If two channels  $\mathcal{N}_1, \mathcal{N}_2$  are composed parallelly then the channel action can be represented by  $\mathcal{N}_1 \otimes \mathcal{N}_2$ . On the other hand if two channels are combined in series then there may be two possibilities in a definite causal structure;  $\mathcal{N}_1 \circ \mathcal{N}_2$  where  $\mathcal{N}_2$  occurs in the causal past of  $\mathcal{N}_1$  and  $\mathcal{N}_2 \circ \mathcal{N}_1$  where  $\mathcal{N}_1$  occurs in the causal past of  $\mathcal{N}_2$ . However, one can use another additional quantum system called *quantum SWITCH* where two channels,  $\mathcal{N}_1$  and

$\mathcal{N}_2$  can take place in a quantum superposition of two alternative orders. Such superposition is controlled by making two alternative orders of  $\mathcal{N}_1$  and  $\mathcal{N}_2$  entangled with a control qubit  $\omega_c = \alpha|0\rangle_c + \beta|1\rangle_c$ . The result is the effective channel,

$$\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2) [\rho \otimes \omega_c] = \sum_{i,j} W_{i,j}(\rho \otimes \omega_c) W_{i,j}^\dagger, \quad (2.34)$$

where  $W_{i,j}$  is the effective Kraus operator after the switch action and is expressed as,

$$W_{i,j} = K_i^{(2)} K_j^{(1)} \otimes |0\rangle_c \langle 0| + K_j^{(1)} K_i^{(2)} \otimes |1\rangle_c \langle 1|. \quad (2.35)$$

Mathematically, *quantum SWITCH* is a supermap or super channel, which takes two channels  $\mathcal{N}_1$  and  $\mathcal{N}_2$  and creates another quantum channel  $\mathcal{S}(\mathcal{N}_1, \mathcal{N}_2)$ . This feature of superposition of alternative orders could emerge in a theory of quantum gravity.

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## DETECTION OF EINSTEIN-PODOLSKY-ROSEN STEERING USING SUM-UNCERTAINTY RELATION

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### 3.1 INTRODUCTION

As mentioned in the previous chapter, Einstein-Podolsky-Rosen-steering (EPR-steering) [4, 11, 66] is an intermediate concept of quantum correlations which lies between Bell nonlocality [7, 8, 43] and entanglement [51]. Apart from the foundational issues, recently, EPR-steering has also received a lot of attention from the perspective of information theoretic applications in semi device-independent scenario. However, before utilizing it in various information processing tasks, at first it is important to detect EPR-steering. The present chapter which is based on one of our works [67], explores one such possibility of detecting the signature of EPR-steerability.

Demonstration of EPR-steering was first proposed by Reid [47] in 1989 as an experimentally testable criterion. However, Wiseman *et al.* [11, 66] formulated a unified information theoretic description of quantum correlations manifested through entanglement, EPR-steering and Bell nonlocality in terms of information theoretic tasks. This paved the way for steering inequalities analogous to Bell inequalities to be formulated to rule out the existence of LHS models and demonstrate steerability. Experimental investigation of EPR-steering has also been performed [48, 49, 68]. Several steering inequalities have since been proposed, with the motivation of obtaining stronger or optimal steering criteria corresponding to particular contexts of the number of parties and measurement settings [69–71]. A necessary and suffi-

cient condition for steering has recently been obtained by Cavalcanti *et al.* [72], in the case of bipartite systems with two measurement settings and two outcomes per party.

The quantum uncertainty principle plays an important role in the manifestation of EPR-steering. For the case of continuous variables, demonstration of EPR-steering by the Reid formalism [47] is based upon the calculation of inferred variances of quadrature amplitudes. Due to correlations in the observables of the two parties sharing an entangled state, the product of inferred variances may fall below the limit obtained through the application of the Heisenberg uncertainty relation (HUR) [73], thus revealing steering. It was subsequently realized that the Reid criterion based on the HUR is incapable of demonstrating steerability of several continuous variable states, notably certain highly entangled and even Bell nonlocal non-Gaussian states. As entropic uncertainty relations (EUR) [74–76] are tighter compared to the HUR, steering inequalities based on EURs [77] are expected to provide stronger steerability conditions. Consequently, it is also shown that entropic steering inequalities are better in revealing steering by non-Gaussian entangled states [78]. A more optimal uncertainty bound is provided by the fine-grained uncertainty relation [22] which has been used to obtain an even tighter fine-grained steering criterion for continuous variables [79].

Recently, variance based sum uncertainty relations [80, 81] have received a lot of attention. Sum uncertainty relations guarantee the lower bound of uncertainty to be non-trivial for incompatible observables whenever the variance of at least one of the observables is non-zero, a feature that is lacking in uncertainty relations based on the product of variances, such as the HUR or the Robertson-Schrödinger uncertainty relation [82]. Sum uncertainty relations therefore, in general, provide a tighter bound of uncertainty compared to product uncertainty relations [83], as has also been realized experimentally [84]. Extensions of the sum uncertainty relations have further been formulated for systems involving multiple observables [85, 86]. In the present context, it is hence imperative to enquire what advantage, if any, would the sum uncertainty relation provide to the corresponding steering criterion, compared to say, an HUR based steering inequality.

With the above motivation, here, a new steering criterion has been derived using an uncertainty relation based on the sum of variances [80]. Our steering criterion is then applied to the case of a bipartite system with two measurements settings and two outcomes. We show

that our steering inequality applied to the Werner state matches the recently derived necessary and sufficient condition for steering [72] for this setting. We then move on to study continuous variable systems where tighter steering conditions based on EURs have been able to reveal steering by several non-Gaussian entangled states [77, 78] in addition to the Gaussian states which mostly admit steering using the standard HUR. Non-gaussian states generally have a higher degree of entanglement compared to Gaussian states, and hence, have applications in tests of Bell inequalities, quantum teleportation, and other quantum information protocols [87–90]. We consider various classes of non-Gaussian states for application of our derived steering inequality. We show that the sum uncertainty based steering criterion improves upon the steering criterion based on HUR and EUR for such states

This chapter is organized as follows. In the next section, a brief overview of various steering criteria derived earlier for discrete and continuous variable systems have been provided. In section (3.3) a steering inequality from an uncertainty relation based on the sum of variances is derived. Next, in section (3.4) it has been shown that our steering condition matches the necessary and sufficient condition for steering in the case of two-qubit Werner states. In section (3.5) our steering inequality is applied on different classes of continuous variable entangled states such as the two mode squeezed vacuum state (TMSV), the photon subtracted TMSV state, the two-dimensional harmonic oscillator state and the results are compared with that of the Reid steering criterion as well as the entropic steering criterion. Finally, section (3.6) is reserved for some concluding remarks.

### 3.2 CONDITIONS FOR EPR-STEERING

Consider an entangled state of two particles, held by two parties (say Alice and Bob)

$$|\Psi\rangle = \sum c_n |\psi_n\rangle |u_n\rangle = \sum d_n |\phi_n\rangle |v_n\rangle, \quad (3.1)$$

where  $|u_n\rangle$  and  $|v_n\rangle$  are two orthonormal bases for Alice's system. If Alice chooses to measure in the  $|u_n\rangle$  ( $|v_n\rangle$ ) basis, she instantaneously projects Bob's system onto one of the states  $|\psi_n\rangle$  ( $|\phi_n\rangle$ ). The steering analogue [72] of the Clauser-Horne-Shimony-Holt (CHSH) inequality provides

a necessary and sufficient criterion for steering in the two measurement per party scenario performed on two-qubit entangled states. The inequality is given as,

$$\sqrt{\langle (A_1 + A_2)B_1 \rangle^2 + \langle (A_1 + A_2)B_2 \rangle^2} + \sqrt{\langle (A_1 - A_2)B_1 \rangle^2 + \langle (A_1 - A_2)B_2 \rangle^2} \leq 2 \quad (3.2)$$

where,  $\{A_1, A_2\}$  are dichotomic measurements on Alice's side and  $\{B_1, B_2\}$  are dichotomic mutually unbiased qubit measurements on Bob's side. the maximum quantum mechanical value of the left hand side is found to be  $2\sqrt{2}$  which can be achieved by Bell states.

For continuous variable systems EPR-steering could be demonstrated using the HUR by showing that the product of variances of inferred values of the correlated observables is less than the lower bound of uncertainty [47]. Defining the quadrature phase amplitudes as

$$\hat{X}_\theta = \frac{\hat{a}e^{-i\theta} + \hat{a}^\dagger e^{i\theta}}{\sqrt{2}}, \hat{Y}_\phi = \frac{\hat{b}e^{-i\phi} + \hat{b}^\dagger e^{i\phi}}{\sqrt{2}} \quad (3.3)$$

where the operators  $\hat{a} = \frac{\hat{X} + i\hat{P}_x}{\sqrt{2}}, \hat{a}^\dagger = \frac{\hat{X} - i\hat{P}_x}{\sqrt{2}}, \hat{b} = \frac{\hat{Y} + i\hat{P}_y}{\sqrt{2}}, \hat{b}^\dagger = \frac{\hat{Y} - i\hat{P}_y}{\sqrt{2}}$  obey the bosonic commutation relations, and hence,  $\hat{X}_\theta = \cos[\theta]\hat{X} + \sin[\theta]\hat{P}_y, \hat{Y}_\phi = \cos[\phi]\hat{Y} + \sin[\phi]\hat{P}_y$ , the correlation function  $C_{\theta,\phi}$  for the quadratures is defined as[47]

$$C_{\theta,\phi} = \frac{\langle \hat{X}_\theta \hat{Y}_\phi \rangle}{\sqrt{\langle \hat{X}_\theta^2 \rangle \langle \hat{Y}_\phi^2 \rangle}} \quad (3.4)$$

In presence of correlations ( $C_{\theta,\phi} \neq 0$ ) the quadrature amplitude  $\hat{X}_\theta$  could be inferred by measuring the corresponding amplitude  $\hat{Y}_\phi$ . Hence, using the HUR

$$\Delta \hat{X}_{\theta_1}^2 \Delta \hat{X}_{\theta_2}^2 \geq 1/4 \quad (3.5)$$

it is possible to derive a bound on the product of variances of the inferred amplitudes, given by [47]

$$(\Delta_{inf} \hat{X}_{\theta_1})^2 (\Delta_{inf} \hat{X}_{\theta_2})^2 \geq 1/4 \quad (3.6)$$

EPR-steering occurs whenever the above inequality is violated by observables acting on some given state.

As stated earlier, it is not possible to reveal steering by several continuous variable entangled states using the Reid inequality (3.6) based on the HUR, even though such states exhibit Bell-violation [77, 78]. A tighter uncertainty bound is provided by the entropic uncertainty relation [74] given by

$$h_Q(X) + h_Q(P) \geq \ln \pi e \quad (3.7)$$

EPR-steering is demonstrated by the non-existence of a LHS model for measurement outcomes. In other words, EPR steering occurs if the joint measurement probability cannot be written as [11, 66]

$$P(r_A, r_B) = \sum_{\lambda} P(\lambda) P(r_A|\lambda) P_Q(r_B|\lambda). \quad (3.8)$$

where  $r_A$  and  $r_B$  are the outcomes of measurements  $R_A$  and  $R_B$ , respectively,  $\lambda$  is the hidden variable, that specifies an ensemble of states,  $P$  are general probability distributions and  $P_Q$  are probability distributions which correspond to measurement on the quantum state specified by  $\lambda$ . The conditional probability  $P(r_B|r_A)$  is given by  $P(r_B|r_A) = \sum_{\lambda} P(r_B, \lambda|r_A)$  (equivalent to above equation) with  $P(r_B, \lambda|r_A) = P(\lambda|r_A) P_Q(r_B|\lambda)$ . Now, using the EUR (3.7), it is possible to derive a correspondingly tighter steering condition given by [77]

$$h(R_B|R_A) + h(S_B|S_A) \geq \ln \pi e \quad (3.9)$$

The violation of the inequality demonstrates steering, as has been explicitly shown for several Gaussian and non-Gaussian entangled states [77, 78].

### 3.3 STEERING INEQUALITY USING SUM-UNCERTAINTY RELATION

We now derive a steering criterion based on the uncertainty bound on the sum of variances of two observables. Let us first consider a typical information theoretic game [11, 66] involving two parties, Alice and Bob. Alice prepares a bipartite quantum system and sends one particle to Bob, and this process can be performed repeatedly. Both the parties can perform

measurements on their respective parts and can communicate classically. Here, Alice's task is to convince Bob that the state they share is entangled. If, on the other hand, Alice tries to cheat by sending a pure state drawn at random from an ensemble to Bob, and chooses her result (communicated to Bob) based on her knowledge of this local hidden state (LHS), the joint probability distribution of their measurement outcomes can be written as [11, 66]

$$P(X_\theta, Y_\phi) = \sum_{\lambda} P(\lambda) P_Q(X_\theta|\lambda) P(Y_\phi|\lambda). \quad (3.10)$$

where  $X_\theta$  is the observable on Bob's side and  $Y_\phi$  is on Alice's side, and where  $P_Q(X_\theta|\lambda)$  represents the probability of  $X_\theta$  predicted by a quantum state  $\rho_\lambda$ .

Our derivation of the steering condition follows the analysis of [47, 69] based on the HUR. When Alice infers the outcomes of Bob's measurement by measuring on her subsystem, the average inference variance of  $X_\theta$  given the estimate  $X_{\theta_{est}}(Y_\phi)$  is defined by

$$\Delta_{inf}^2 X_\theta = \langle [X_\theta - X_{\theta_{est}}(Y_\phi)]^2 \rangle, \quad (3.11)$$

where  $X_{\theta_{est}}(Y_\phi)$  is Alice's estimate of the value of Bob's measurement  $X_\theta$  as a function of her measurement outcome  $Y_\phi$ , and the average is over all outcomes. The estimate that minimizes the r.h.s. of the above equation is for  $X_{\theta_{est}}(Y_\phi) = \langle X_\theta \rangle_{Y_\phi}$  [69]. Thus, the optimal inference variance of  $X_\theta$  by measurement of  $Y_\phi$  is given by

$$\begin{aligned} \Delta_{min}^2 X_\theta &= \langle [X_\theta - \langle X_\theta \rangle_{Y_\phi}]^2 \rangle \\ &= \sum_{X_\theta, Y_\phi} P(X_\theta, Y_\phi) (X_\theta - \langle X_\theta \rangle_{Y_\phi})^2 \\ &= \sum_{Y_\phi} P(Y_\phi) \sum_{X_\theta} P(X_\theta|Y_\phi) (X_\theta - \langle X_\theta \rangle_{Y_\phi})^2 \\ &= \sum_{Y_\phi} P(Y_\phi) \Delta^2(X_\theta|Y_\phi), \end{aligned} \quad (3.12)$$

where  $\Delta^2(X_\theta|Y_\phi)$  is the variance of  $X_\theta$  calculated from the conditional probability distribution  $P(X_\theta|Y_\phi)$ , and by definition

$$\Delta_{inf}^2 X_\theta \geq \Delta_{min}^2 X_\theta. \quad (3.13)$$

Assuming the LHS model given by equation (3.10), the conditional probability of  $X_\theta$  given  $Y_\phi$  can be written as

$$\begin{aligned} P(X_\theta|Y_\phi) &= \sum_{\lambda} \frac{P(\lambda)P(Y_\phi|\lambda)}{P(Y_\phi)} P_Q(X_\theta|\lambda) \\ &= \sum_{\lambda} P(\lambda|Y_\phi) P_Q(X_\theta|\lambda) \end{aligned} \quad (3.14)$$

Since  $P(x)$  has a convex decomposition [ $P(x) = \sum_y P(y)P(x|y)$ ], the variance  $\Delta^2 x$  over the distribution  $P(x)$  cannot be smaller than the average of the variances over the component distribution  $P(x|y)$ , i.e.,  $\Delta^2 x \geq \sum_y P(y)\Delta^2(x|y)$  [69]. Then, from the above equation, the variance  $\Delta^2(X_\theta|Y_\phi)$  satisfies

$$\Delta^2(X_\theta|Y_\phi) \geq \sum_{\lambda} P(\lambda|Y_\phi)\Delta_Q^2(X_\theta|\lambda), \quad (3.15)$$

where  $\Delta_Q^2(X_\theta|\lambda)$  is the variance of  $P_Q(X_\theta|\lambda)$ . From the above result it follows that the bound for  $\Delta_{min}^2 X_\theta$  is given by

$$\begin{aligned} \Delta_{min}^2 X_\theta &\geq \sum_{Y_\phi, \lambda} P(Y_\phi, \lambda)\Delta_Q^2(X_\theta|\lambda) \\ &= \sum_{\lambda} P(\lambda)\Delta_Q^2(X_\theta|\lambda). \end{aligned} \quad (3.16)$$

Hence, for two variables on Bob's side, say  $X_{\theta_1}$  and  $X_{\theta_2}$  using Eqs.(3.13) and (3.16) one has,

$$\Delta_{inf}^2 X_{\theta_1} \geq \sum_{\lambda} P(\lambda)\Delta_Q^2(X_{\theta_1}|\lambda), \quad \Delta_{inf}^2 X_{\theta_2} \geq \sum_{\lambda} P(\lambda)\Delta_Q^2(X_{\theta_2}|\lambda)$$

or in general

$$\Delta_{inf}^2 X_{\theta_i} \geq \sum_{\lambda} P(\lambda)\Delta_Q^2(X_{\theta_i}|\lambda). \quad (3.17)$$

Now, let us define two vectors  $u$  and  $v$  s.t.  $u \equiv [u_1, u_2, \dots, u_i, \dots]$ , where  $u_i = \sqrt{P(\lambda_i)}\Delta_Q(X_{\theta_1}|\lambda_i)$ , are the components of the vector  $u$ , and similarly,  $v \equiv [v_1, v_2, \dots, v_i, \dots]$  with components  $v_i = \sqrt{P(\lambda_i)}\Delta_Q(X_{\theta_2}|\lambda_i)$ . Noting that  $\Delta_{inf}^2 X_{\theta_1} \geq |u|$  ( $\equiv \sqrt{u_1^2 + u_2^2 + \dots}$ ), and similarly, for  $v$ , in terms of  $u$  and  $v$ , it follows from Eq.(3.17) that  $\Delta_{inf}^2 X_{\theta_1} \geq |u|$  and  $\Delta_{inf}^2 X_{\theta_2} \geq |v|$ .

Using the triangle inequality ( $|u| + |v| \geq |u + v|$ ) one thus obtains  $\Delta_{inf} X_{\theta_1} + \Delta_{inf} X_{\theta_2} \geq \sqrt{(u_1 + v_1)^2 + (u_2 + v_2)^2 + \dots}$ , and hence in summation form,

$$\sum_{i=1}^2 \Delta_{inf} X_{\theta_i} \geq \sqrt{\sum_{\lambda} P(\lambda) \left[ \sum_{i=1}^2 \Delta_Q(X_{\theta_i} | \lambda) \right]^2}. \quad (3.18)$$

It is known that the quantum fluctuation in the sum of any two observables is always less than or equal to the sum of their individual fluctuations, [80], i.e.,

$$\Delta(A_1 + A_2) \leq \Delta A_1 + \Delta A_2. \quad (3.19)$$

Using the above uncertainty relation (3.19) in the right hand side of Eq.(3.18), one gets

$$\sum_{i=1}^2 \Delta_{inf} X_{\theta_i} \geq \sqrt{\sum_{\lambda} P(\lambda) \left[ \Delta_Q\left(\sum_{i=1}^2 X_{\theta_i}\right) | \lambda \right]^2}. \quad (3.20)$$

Since we have assumed a LHS model for Bob, the right hand side of the above equation therefore corresponds to the variance of the sum of the observables  $X_{\theta_1}$ , and  $X_{\theta_2}$ , i.e.,  $\Delta(X_{\theta_1} + X_{\theta_2})$ . We thus get the sum-steering inequality given by

$$\Delta(X_{\theta_1} + X_{\theta_2}) \leq \Delta_{inf} X_{\theta_1} + \Delta_{inf} X_{\theta_2}. \quad (3.21)$$

A violation of this inequality will demonstrate steering. It may be noted that the variance of the measured observables on each individual side will satisfy the uncertainty relation  $\Delta(X_{\theta_1} + X_{\theta_2}) \leq \Delta X_{\theta_1} + \Delta X_{\theta_2}$ . But, due to the presence of correlations, Alice's measurement of  $Y_{\phi}$  may be used to infer the value of  $X_{\theta}$  on Bob's side. Steering takes place if the calculated uncertainties for the inferred observables violate Eq.(3.21). In other words, if the value of  $\Delta_{inf}(X_1) + \Delta_{inf}(X_2)$  becomes less than that lower bound of Eq.(3.21), we can say that the sum uncertainty relation is able to reveal steering. This is our steering criterion.

### 3.4 SUM-STEERING CONDITION FOR TWO QUBIT WERNER STATES

To illustrate the EPR steering criterion given by Eq.(3.21) for the case of discrete variables, consider the two-qubit Werner state [91]

$$\rho_W = p|\psi^-\rangle\langle\psi^-| + \frac{1-p}{4}\mathbb{I} \quad (3.22)$$

where,  $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$  is the singlet state corresponding to  $\sigma_z$ -eigenbasis,  $\{|0\rangle, |1\rangle\}$ , and  $\frac{\mathbb{I}}{4}$  is the maximally mixed state with  $0 \leq p \leq 1$ .

Corresponding to two non-commuting spin- $\frac{1}{2}$  observables  $\{S_x, S_z\}$ , ( $S_i = \frac{\sigma_i}{2}$ ) on Bob's side, the sum-steering inequality( 3.21) looks like,

$$\Delta(S_x^B + S_z^B) \leq \Delta_{\text{inf}}S_x^B + \Delta_{\text{inf}}S_z^B. \quad (3.23)$$

The inferred values for the observables  $\{S_x, S_z\}$  can be calculated using the Reid prescription [47] using the correlation function. For two general observables on each side with the correlation function defined as  $C_{ij} = \frac{\langle A_i B_j \rangle}{\sqrt{\langle A_i^2 \rangle \langle B_j^2 \rangle}}$ , the estimates of the variables  $B_1$  and  $B_2$  on Bob's side are given in terms of measurements of Alice's variables  $A_1$  and  $A_2$ , by  $B_1^e = g_1 A_1$  and  $B_2^e = g_2 A_2$ , where  $g_1$  and  $g_2$  correspond to the errors in estimation. The average errors of inference are given by  $(\Delta_{\text{inf}}A_1)^2 = \langle (B_1 - B_1^e)^2 \rangle = \langle (B_1 - g_1 A_1)^2 \rangle$ , and similarly for  $(\Delta_{\text{inf}}A_2)^2$ . Extremization of the inferred errors leads to the conditions  $g_1 = \frac{\langle A_1 B_1 \rangle}{\langle A_1^2 \rangle}$  and  $g_2 = \frac{\langle A_2 B_2 \rangle}{\langle A_2^2 \rangle}$ , which are plugged back into the expressions for the inferred observables to yield the inferred variances  $(\Delta_{\text{inf}}B_1)^2$  and  $(\Delta_{\text{inf}}B_2)^2$ .

For the case of the particular observables chosen here, it can be simplified that,  $\Delta_{\text{inf}}S_x^B = \Delta_{\text{inf}}S_z^B = \frac{\sqrt{1-p^2}}{2}$ . Further, one can calculate that  $\Delta(S_x^B + S_z^B) = \frac{1}{\sqrt{2}}$ . Using these results, it is now obvious that the sum-steering inequality (3.23) is violated for  $p > \frac{1}{\sqrt{2}}$  which is optimal for any EPR-steering inequality in the two-measurement setting. Fig. 3.1 depicts the fact that for this range of  $p$ , the right hand side of Eq.(3.23) becomes less than the lower bound

corresponding to the left hand side. For comparison, the entropic steering inequality [70, 76, 92] is also plotted. The left hand side of the entropic steering inequality becomes

$$H(\sigma_x^B|\sigma_x^A) + H(\sigma_z^B|\sigma_z^A) = -\{(1+p)\log[(1+p)/2] + (1-p)\log[(1-p)/2]\}. \quad (3.24)$$

The above inequality is violated for  $p > 0.78$ , indicating that the entropic steering criterion is not an optimal one. There exist however, other two-measurement steering inequalities [68,

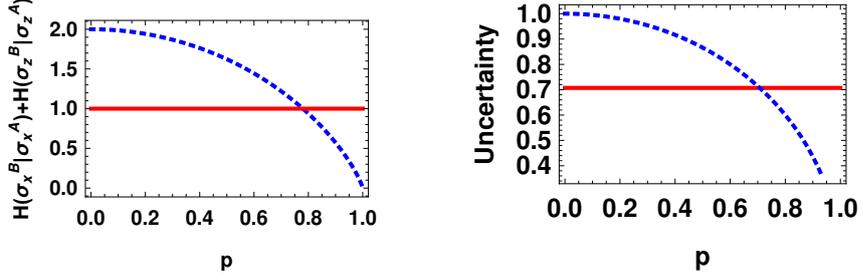


Figure 3.1: The left panel displays the entropic steering inequality for spin-1/2 observables on the Werner state. The blue curve mimics Eq.(3.24) while the red curve denotes the lower bound of the steering condition. The right panel displays violation of the sum uncertainty bound by the right hand side of Eq.(3.23), which occurs for  $p > \frac{1}{\sqrt{2}}$ .

69] that are violated by Werner states in parameter range,  $p > \frac{1}{\sqrt{2}}$ . The two-measurement steering analogue of the CHSH inequality [72] given by Eq.(3.2) provides a necessary and sufficient criterion for steering in the two measurement per party scenario performed on two-qubit entangled states. This steering inequality demonstrates steering iff  $p > \frac{1}{\sqrt{2}}$  [93]. Thus, the optimal range of steerability captured by the inequality given by Eq.(3.2) in the discrete variable two-measurement setting is similar to that obtained using the sum-steering inequality (3.21) derived by us.

### 3.5 EPR-STEERING FOR CONTINUOUS VARIABLE SYSTEMS

In this section, the steerability of various continuous variable states have been studied using our sum-steering relation. The comparison among our sum-steering criterion with that of the Reid and entropic steering inequalities are also been explored. We first consider the two mode squeezed vacuum state, and then two examples of non-Gaussian entangled states, *viz.*

the photon subtracted squeezed vacuum state, and the two-dimensional harmonic oscillator given in terms of Laguerre-Gaussian (LG) wave functions. We calculate the magnitude of violation of our steering inequality and compare it with that obtained from the Reid and entropic steering inequalities.

### 3.5.1 Two mode squeezed vacuum

The two mode squeezed vacuum state can be generated by applying the two mode squeezing operator  $S(\xi) = e^{\xi \hat{a}_1^\dagger \hat{a}_2^\dagger - \xi^* \hat{a}_1 \hat{a}_2}$ , (where  $\xi = r e^{i\phi}$ ) on the two mode vacuum state  $|0,0\rangle$ , and is given by

$$|NOPA\rangle = |\xi\rangle = S(\xi)|0,0\rangle = \sqrt{1 - \lambda^2} \sum_{n=0}^{\infty} \lambda^n |n, n\rangle, \quad (3.25)$$

where  $\lambda = \tanh(r) \in [0,1]$ , and the squeezing parameter  $r > 0$ . The Wigner function corresponding to above state is given by [78, 94]

$$W_{\xi}(X, P_X, Y, P_Y) = \frac{1}{\pi^2} \exp[-2(P_X P_Y - XY) \sinh 2r - (X^2 + Y^2 + P_X^2 + P_Y^2) \cosh 2r]. \quad (3.26)$$

The inferred uncertainty is calculated to be [47, 78],

$$(\Delta_{inf} X_{\theta})^2 = \frac{1}{2} \cosh[2r] - \frac{1}{2} \tanh[2r] \sinh[2r] \cos^2[\theta + \phi]. \quad (3.27)$$

We calculate  $(\Delta_{inf} X_{\theta_1})^2$  and  $(\Delta_{inf} X_{\theta_2})^2$  with the settings  $\theta_1 = 0$ ,  $\theta_2 = \pi/2$ ,  $\phi_1 = 0$  and  $\phi_2 = \pi/2$  (the correlations  $\langle XY \rangle$  and  $\langle P_X P_Y \rangle$  are maximized), and hence one obtains

$$(\Delta_{inf} X_{\theta_1})^2 = (\Delta_{inf} X_{\theta_2})^2 = \frac{1}{2 \cosh[2r]}. \quad (3.28)$$

Thus, the product of uncertainties  $\frac{1}{4 \cosh^2[2r]}$  is always less than the uncertainty bound (1/4) [47] (for  $r > 0$ ). The Reid criterion (3.6) is able to show steering for such states for all squeezing

parameters. One can also apply the entropic steering inequality (3.9) for this state. Since the non-vanishing correlations are  $\langle XY \rangle$  and  $\langle P_X P_Y \rangle$ , the inequality becomes [78],

$$h(\mathcal{X}|\mathcal{Y}) + h(\mathcal{P}_X|\mathcal{P}_Y) \geq \ln \pi e. \quad (3.29)$$

The conditional entropies are given by  $h(\mathcal{X}|\mathcal{Y}) = h(\mathcal{X}, \mathcal{Y}) - h(\mathcal{Y})$ ,  $h(\mathcal{P}_X|\mathcal{P}_Y) = h(\mathcal{P}_X, \mathcal{P}_Y) - h(\mathcal{P}_Y)$  with  $h(\mathcal{X}, \mathcal{Y}) = - \int_{-\infty}^{\infty} P(X, Y) \ln P(X, Y) dXdY$ , and similarly for the other entropies. The probability distributions are obtained from the Wigner function (3.26). It is already known that entropic uncertainty relation is also able to show steering for all  $r$  [78]. In or-

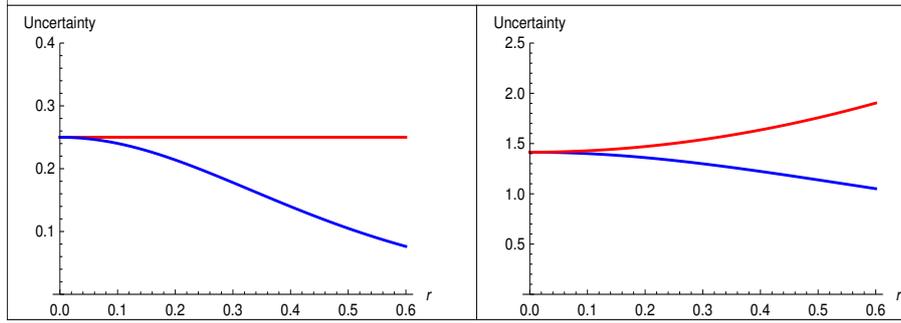


Figure 3.2: The left panel shows that the product of inferred uncertainties for the two mode squeezed vacuum state given by the blue curve falls below the bound (red line) obtained from the Reid inequality. The right panel shows that the sum of inferred uncertainties (blue) falls below the bound (3.30) (red) obtained using our sum-steering relation.

der to apply our sum-steering criterion, let us now calculate  $\Delta_{inf}(X_{\theta_1}) + \Delta_{inf}(X_{\theta_2})$  and the uncertainty bound,  $\Delta(X_{\theta_1} + X_{\theta_2})$ . We get,

$$\begin{aligned} \Delta_{inf}(X_{\theta_1}) + \Delta_{inf}(X_{\theta_2}) &= \sqrt{\frac{2}{\cosh[2r]}}, \\ \Delta(X_{\theta_1} + X_{\theta_2}) &= \sqrt{2 \cosh[2r]}. \end{aligned} \quad (3.30)$$

The uncertainty bound and the inferred uncertainty are plotted in Fig. 3.2. It is clear from the plot that our steerability criterion based on sum uncertainty relation is able to demonstrate steerability of the two mode squeezed vacuum state for the whole range of squeezing parameter ' $r$ '.

### 3.5.2 Single photon-subtracted squeezed vacuum

A non-Gaussian state derived from a two-mode squeezed vacuum by subtracting a single photon from any of the two modes may be written as

$$|\xi_{-1}\rangle = \frac{1}{2 \sinh^2(r)} \sqrt{1 - \lambda^2} \sum_{n=0}^{\infty} \lambda^n \sqrt{n} [ |n-1, n\rangle + (-1)^k |n, n-1\rangle ], \quad (3.31)$$

with  $k \in [0, 1]$ . The Wigner function corresponding to this single-photon subtracted state in terms of  $X, P_X, Y, P_Y$  can be calculated from the Wigner function of the two-mode squeezed vacuum state [94], given by

$$W_1(X, Y, P_X, P_Y) = \frac{1}{\pi^2} \exp \left[ 2 \sinh(2r)(XY - P_X P_Y) - \cosh(2r)(X^2 + Y^2 + P_X^2 + P_Y^2) \right] \times \\ [-\sinh(2r)[P_X^2 - 2P_X P_Y + P_Y^2 - (X - Y)^2] + \cosh(2r)[P_X^2 - 2P_X P_Y + P_Y^2 + (X - Y)^2] - 1].$$

The uncertainties for the inferred observables  $X_{\theta_1}$  and  $X_{\theta_2}$  (for conjugate variables we take  $\theta_1 = 0$  and  $\theta_2 = \pi/2$ ) are given by

$$(\Delta_{inf} X_{\theta_1})^2 = \frac{3}{[4 \{ \cosh(2r) - \cosh(r) \cdot \sinh(r) \}]} \quad (3.32)$$

and

$$(\Delta_{inf} X_{\theta_2})^2 = \frac{3}{[4 \{ \cosh(2r) + \cosh(r) \cdot \sinh(r) \}]} \quad (3.33)$$

leading to the product of the uncertainties

$$(\Delta_{inf} X_{\theta_1})^2 (\Delta_{inf} X_{\theta_2})^2 = \frac{9}{2[3 \cosh(4r) + 5]}. \quad (3.34)$$

A plot of the product of inferred uncertainties and the uncertainty bound given by the Reid criterion with respect to the squeezing parameter ' $r$ ' is provided in Fig. 3.3. It is clear from the plot that the Reid inequality fails to exhibit steering for smaller value of  $r$ , as already known in the literature [78]. However, the entropic steering inequality (3.29) is able to reveal the steerability of this state, as shown earlier [78].

Now, in order to check steering using the sum uncertainty relation, the uncertainty bound for the photon-subtracted state is calculated according to sum-uncertainty relation,

$$\Delta(X_{\theta_1} + X_{\theta_2}) = \sqrt{\cosh[2r] - \cos[r] \sinh[r]} + \sqrt{\cosh[2r] + \cos[r] \sinh[r]}. \quad (3.35)$$

Due to the presence of correlations  $\langle XY \rangle$  and  $\langle P_X P_Y \rangle$ , the sum of inferred uncertainties, calculated using Wigner function turn out to be

$$(\Delta_{inf} X_{\theta_1}) + (\Delta_{inf} X_{\theta_2}) = \sqrt{\frac{3}{4[\cosh[2r] - \cosh[r] \sinh[r]]}} + \sqrt{\frac{3}{4[\cosh[2r] + \cosh[r] \sinh[r]]}}. \quad (3.36)$$

The sum uncertainty bound and the sum of inferred uncertainties for the single photon subtracted squeezed vacuum state are plotted in Fig. 3.3. Clearly, our steering criterion is able to show steering for all values of the squeezing parameter ' $r$ '.

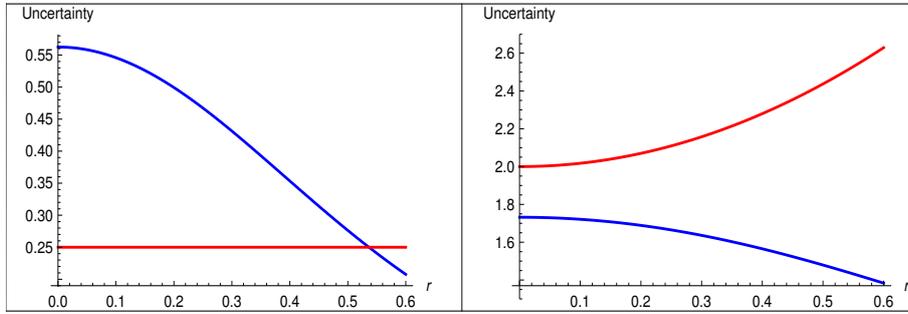


Figure 3.3: Steering by the single photon subtracted squeezed vacuum state. On the left hand panel the product of inferred uncertainties (the blue curve), and the Reid bound (red) are plotted versus the squeezing parameter  $r$ . The right hand panel depicts the sum of inferred uncertainties (the blue curve) and sum-steering bound (the red curve). It is clear that the steering inequality based on sum uncertainty relation is able to show steering for all values of the squeezing parameter.

Next, we provide a comparison of the magnitude of steering by the three different criteria, *viz.*, the Reid criterion, the entropic steering relation, and the sum-steering relation for the single photon subtracted squeezed vacuum state. The magnitude of violation for all these steering criteria are presented in the table (3.1), as functions of the squeezing parameter  $r$ . It is clear from the table that the magnitude of violation of the sum-steering inequality is always greater than both the Reid and the entropic steering inequalities.

$r$	$\frac{1}{4(\Delta_{inf} \bar{x}_{\theta_1})^2 \Delta_{inf} \bar{x}_{\theta_1}}^2$	$\frac{\ln \pi e}{h(\mathcal{X} \mathcal{P}_Y) + h(\mathcal{Y} \mathcal{P}_X)}$	$\frac{\Delta(\mathcal{X}_1 + \mathcal{X}_2)}{\Delta_{inf}(\mathcal{X}_1) + \Delta_{inf}(\mathcal{X}_2)}$
0	0.444	1.044	1.155
0.1	0.458	1.053	1.161
0.2	0.501	1.061	1.225
0.3	0.581	1.093	1.318
0.4	0.707	1.124	1.457
0.5	0.909	1.192	1.648
0.6	1.204	1.264	1.901

Table 3.1: Comparison of the magnitude of steering by the three different criteria, *viz.*, the Reid criterion, the entropic steering relation, and the sum-steering relation for the single photon subtracted squeezed vacuum state.

### 3.5.3 Two-dimensional harmonic oscillator states

For the two-dimensional harmonic oscillator, the wave-function in terms of the Hermite-Gauss function is given by [95]

$$u_{mn}(x, y) = \sqrt{\frac{2}{\pi}} \left\{ \frac{1}{2^{m+n} w^2 m! n!} \right\}^{\frac{1}{2}} H_m\left(\frac{\sqrt{2}x}{w}\right) H_n\left(\frac{\sqrt{2}y}{w}\right) \times \exp[-(x^2 + y^2)/w^2]. \quad (3.37)$$

It is possible to construct entangled states using superpositions of the above Hermite-Gaussian wave functions, that can be represented by Laguerre-Gaussian (LG) beams given by [94],

$$\Phi_{mn}(\rho, \theta) = e^{-\rho^2/w^2} e^{i(m-n)\theta} (-1)^{\min(m,n)} \left(\frac{\rho\sqrt{2}}{w}\right)^{|m-n|} \sqrt{\frac{2}{\pi m! n! w^2}} L_{\min(m,n)}^{|m-n|} \left(\frac{2\rho^2}{w^2}\right) [\min(m, n)]!, \quad (3.38)$$

written in terms of cylindrical coordinates using the generalised Laguerre polynomial. The Wigner function corresponding to the LG-beam in terms of the dimensionless quadratures  $X, P_X, Y, P_Y$  is given by [78, 96]

$$W_{m,n}(X, P_X, Y, P_Y) = \frac{(-1)^{(m+n)}}{\pi^2} L_m[4(Q_0 + Q_2)] L_n[4(Q_0 - Q_2)] \exp(-4Q_0), \quad (3.39)$$

where  $Q_0 = \frac{1}{4}[X^2 + Y^2 + P_X^2 + P_Y^2]$ , and  $Q_2 = \frac{XP_Y - YP_X}{2}$ . It was shown earlier [78] that the Reid criterion fails to demonstrate steering for LG-beams. However, the entropic steering criterion is able to reveal steerability of LG-beams for all values of  $n \geq 1$ .

Let us now apply our sum-steering inequality (3.21) for the case of LG-beams. For this, the uncertainty bound as well as the inferred uncertainty for the LG-beams need to be computed. The sum uncertainty bound is obtained in terms of the quadratures, i.e.,

$$\Delta(X_{\theta_1} + X_{\theta_2}) = \Delta(X + P_X) = \sqrt{\langle (X + P_X)^2 \rangle - \langle X + P_X \rangle^2}, \quad (3.40)$$

and, similarly for the inferred variances

$$(\Delta_{inf} X_{\theta_1})^2 = (\Delta_{inf} X)^2 = \langle X^2 \rangle [1 - (C_{0,\pi/2}^{max})^2] = \langle X^2 \rangle \left[ 1 - \frac{\langle XP_Y \rangle}{\sqrt{\langle X^2 \rangle \langle P_Y^2 \rangle}} \right],$$

and

$$(\Delta_{inf} X_{\theta_2})^2 = (\Delta_{inf} P_X)^2 = \langle P_X^2 \rangle [1 - (C_{0,\pi/2}^{max})^2] = \langle P_X^2 \rangle \left[ 1 - \frac{\langle P_X Y \rangle}{\sqrt{\langle P_X^2 \rangle \langle Y^2 \rangle}} \right].$$

The computed values of the above variables are plotted in Fig. 3.4 for several values of the beam angular momentum  $n$ , (taking  $m = 0$ ). It is clear that steering is demonstrated for all  $n \geq 1$ . Additionally, the violation of the sum-steering inequality becomes stronger for higher  $n$ , a feature that is absent in regard to the violation of the entropic steering inequality for LG beams [78]. In the table (3.2) we compare the magnitude of violation of our sum-steering inequality with that of the Reid inequality and the entropic steering inequality. It is clear from the table that not only does the sum-steering relation perform better than the entropic steering inequality for any particular value of  $n$  (there is no violation of the Reid inequality), but also that magnitude of violation of the sum-steering inequality gets stronger with higher angular momentum.

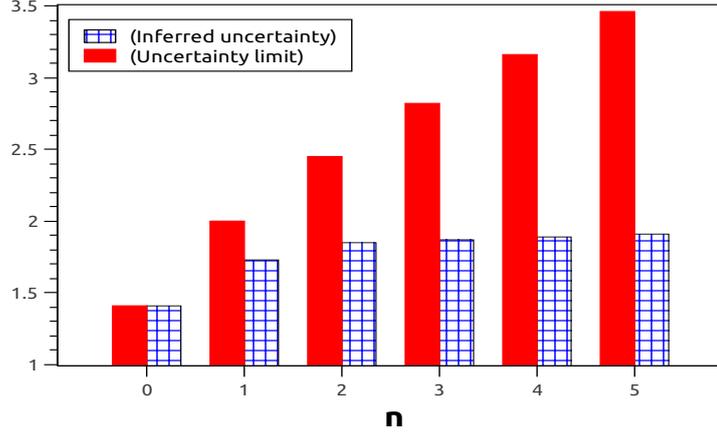


Figure 3.4: The sum uncertainty bound (red and solid bar) and the sum of inferred uncertainties (blue and meshed bar) are plotted versus angular momentum  $n$  for the LG beams. The figure shows clearly that the violation of the sum-steering inequality increases with larger angular momentum.

$n$	$\frac{1}{4(\Delta_{inf} \bar{X}_{\theta_1})^2 \Delta_{inf} \bar{X}_{\theta_2})^2}$	$\frac{\ln \pi e}{h(\mathcal{X} \mathcal{P}_Y) + h(\mathcal{Y} \mathcal{P}_X)}$	$\frac{\Delta(X_1 + X_2)}{\Delta_{inf}(X_1) + \Delta_{inf}(X_2)}$
0	1	1	1
1	0.4444	1.0438	1.1560
2	0.3599	1.0567	1.3243
3	0.3265	1.0626	1.5080
4	0.3086	1.0657	1.6719
5	0.2975	1.0676	1.8115

Table 3.2: Comparison of the magnitude of steering by the three different criteria, *viz.*, the Reid criterion, the entropic steering relation, and the sum-steering relation for the LG-beams.

### 3.6 SUMMARY AND OUTLOOK

In summary, in this work, a new steering inequality has been derived using the sum uncertainty relation [80, 81]. Our derived steering criterion is based on the sum of inferred variances pertaining to two observables of a bipartite system. Since, sum uncertainty relation provides a tighter uncertainty bound compared to the product (Heisenberg) uncertainty relation, the resultant steering inequality based on the former is expected to yield a tighter steering relation compared to that obtained from the the latter [47, 68]. In the context of discrete variables, the sum-steering relation is able to replicate the steering range of Werner states obtained using the necessary and sufficient condition of steerability for the case of

bipartite systems with two measurement settings and two outcomes [72]. Application of the sum-uncertainty based steering relation for continuous variable systems demonstrates its advantage over other approaches based on the Reid criterion [47] and the entropic steering criterion [77]. More specifically, considering some examples of non-Gaussian states such as the photon subtracted squeezed vacuum state and the two-dimensional harmonic oscillator state we obtain stronger violations of the sum-steering inequality compared to those obtained using the Reid inequality as well as the entropic steering inequality [78]. The sum uncertainty based steering relation thus offers a better prospect of detection of steerability compared to other steering criteria for continuous variable systems. It would thus be interesting to explore the practical feasibility of one-sided device independent key generation [97, 98] schemes based on the sum-steering relation for continuous variable entangled states.

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## SEQUENTIAL DETECTION OF GENUINE TRIPARTITE ENTANGLEMENT

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### 4.1 INTRODUCTION

As mentioned in the chapter 1, Entanglement [51] is one of the most fascinating non-classical features of quantum mechanics. The demarcation between separable and entangled states is well understood in the bipartite scenario. But the situation becomes complex in multipartite scenarios as one can consider entanglement across many possible bipartitions. In chapter 2, the notion of genuine entanglement [53] has been introduced which may arise only in the context of multipartite scenario. A multipartite state is called genuinely entangled iff it is not separable with respect to any partition. The concept of genuine entanglement is not only important for quantum foundational research, but also finds various information theoretic implications, for example, in extreme spin squeezing [99], high sensitive metrology tasks [100, 101], quantum computation using cluster states [102], measurement-based quantum computation [103] and multiparty quantum networks [104–107].

In spite of various successful attempts for the generation and detection of genuine multipartite entangled states [108–110], the complication of the process is appreciated as the detection or verification of entanglement involves tomography or constructions of entanglement witnesses under precise experimental control over the system subjected to measurements. Due the difficulties present in generating genuine entanglement which is the resource for a vast range of information processing tasks, it is a significant question to ask whether genuine entanglement can be preserved partially even after performing a few cycles of local

operations. The present chapter which is based on one of our paper [111] address the above question in the tripartite scenario.

The general question as to what extent quantum correlation of an entangled state can be shared by multiple observers who perform measurements sequentially and independently of each other, was first posed in the case of the bipartite scenario. Silva *et al.* [112] addressed this question in the context of Bell nonlocality [7, 8] by considering a scenario where an entangled pair of two spin- $\frac{1}{2}$  particles are shared between Alice in one wing and multiple Bobs in another wing. Alice acts on the first particle and multiple Bobs act on the second particle sequentially, where Alice is spatially separated from the multiple Bobs. In this scenario, using a measurement model, which optimizes the trade-off between information gain and disturbance, it was conjectured [112] that at most two Bobs can violate the Bell-CHSH (Bell-Clauser-Horne-Shimony-Holt) inequality [7, 8] with a single Alice. This result is valid when the choice of measurement settings of each Bob is independent of the choices of measurement settings and outcomes of the previous Bobs and the frequencies of the inputs of each Bob are the same. This result that was subsequently confirmed analytically [113] applying a one-parameter positive operator valued measurement (POVM) [114, 115].

Various experiments have been performed to demonstrate this phenomena [116, 117]. Recently, the notion of shareability of quantum nonlocality has been extended to investigate several other kinds of quantum correlations. These include sharing of EPR steering [118, 119], entanglement [120, 121], steerability of local quantum coherence [122], Bell nonlocality with respect to quantum violations of various other Bell type inequalities [123], and preparation contextuality [124]. These ideas have been applied in randomness generation [125], their classical communication cost [126], quantum teleportation [127], and random access codes [128].

Most of the previous studies have addressed the issue of sharing quantum correlations by multiple sequential observers in the bipartite scenario. Very recently, the possibility of sequential sharing of genuine tripartite nonlocality by multiple observers has been studied [129]. Quantum entanglement is the primary ingredient for nonlocal correlations, and in the present study we focus our attention on the sharing of genuine multipartite entanglement. In particular, we consider the scenario where three spin- $\frac{1}{2}$  particles are spatially separated and

shared between, say, Alice, Bob and multiple Charlies. Alice measures on the first particle; Bob measures on the second particle and multiple Charlies measure on the third particle sequentially. In this scenario, we investigate how many Charlies can detect genuine tripartite entanglement.

In order to detect entanglement, one may consider the violation of Bell-type inequalities as a criterion, since entanglement is a necessary resource for generating nonlocal correlations. One can construct inequalities which can certify genuine multipartite entanglement from the statistical data alone. This method of device-independent detection of genuine entanglement was first introduced in [130–133] followed by an extensive formalization by Bancal *et al.* [134]. Pal [135] and Liang *et al.* [136] have improved the existing inequalities for detecting genuine multipartite entanglement. The Mermin polynomial [137] which is a useful tool for device-independent entanglement-witness can be used to detect genuine tripartite entanglement [130]. In the present study, we use quantum violations of the Mermin inequality [137] and the Uffink inequality [132], respectively, in order to probe detection of genuine tripartite entanglement by multiple sequential Charlies.

Another well developed tool for detection of entanglement is through the entanglement witness operators [52, 138–142]. For each entangled state, there always exists a witness operator which is a consequence of the Hahn-Banach theorem [143]. A similar concept has been formulated for the genuine tripartite entangled states (W-state and GHZ-state) which distinguishes genuine entanglement from the set of all bi-separable states [53, 144–146]. Here, we further analyse the idea of sequential detection of genuine tripartite entanglement using appropriate witness operators.

All our analyses point out that it is indeed possible to detect genuine entanglement sequentially by more than one Charlie. In particular, we show that at most two Charlies can detect genuine entanglement sequentially using the linear as well as nonlinear device-independent genuine entanglement inequalities. On the other hand, through appropriate genuine entanglement witnesses which are suitable for the W-state and the GHZ-state, at most four Charlies and twelve Charlies can respectively, detect genuine entanglement. Hence, this result can be useful in recycling genuine multipartite entangled resources in the context of various information processing tasks.

The chapter is organized as follows: in Section 4.2 we present the basic tools for detecting genuine tripartite entanglement. The measurement scenario involving multiple sequential observers is also described in this Section. In Section 4.3, we present the main results of this work, namely, sequential detection of genuine tripartite entanglement. Finally, we conclude in Section 4.5.

## 4.2 PRELIMINARIES

In this Section we will present some basic tools which will be used throughout this chapter. We will also elaborate on the scenario in which sequential detection of genuine tripartite entanglement is studied.

### 4.2.1 Detection of Genuine Entanglement

In order to certify genuine entanglement in a device-independent way, several inequalities have been proposed. For the purpose of the present work, we will use some of them. As mentioned in chapter 2, a tripartite state  $\rho$  is said to be genuinely entangled if and only if the state can not be written in the following bi-separable form,

$$\rho = \sum_{\lambda} p_{\lambda} \rho_{\lambda}^A \otimes \rho_{\lambda}^{BC} + \sum_{\mu} p_{\mu} \rho_{\mu}^B \otimes \rho_{\mu}^{AC} + \sum_{\nu} p_{\nu} \rho_{\nu}^C \otimes \rho_{\nu}^{AB}, \quad (4.1)$$

with  $0 \leq p_{\lambda}, p_{\mu}, p_{\nu} \leq 1$  and  $\sum_{\lambda} p_{\lambda} + \sum_{\mu} p_{\mu} + \sum_{\nu} p_{\nu} = 1$ .

Let us begin with presenting the device-independent entanglement-witness provided by the Mermin polynomial [137] as the simplest example for detecting genuine tripartite entanglement [130]. Consider that three spatially separated parties, say, Alice, Bob and Charlie are sharing some quantum system in the state  $\rho$ . The choices of measurement settings, performed by Alice, Bob and Charlie on the shared state  $\rho$  are denoted by  $A_x$ ,  $B_y$  and  $C_z$  respectively, where  $x, y, z \in \{0, 1\}$ . The outcomes of Alice, Bob and Charlie's measurements are denoted by  $a, b$  and  $c$ , respectively, with  $a, b, c \in \{+1, -1\}$ . By repeating the experiment a number of times, the joint probability distributions  $P(a, b, c|x, y, z)$  are produced. In this sce-

nario, the Mermin inequality, whose violation certifies the presence of genuine entanglement in a device-independent way, can be expressed as [130, 147]:

$$M = |\langle A_1 B_0 C_0 \rangle + \langle A_0 B_1 C_0 \rangle + \langle A_0 B_0 C_1 \rangle - \langle A_1 B_1 C_1 \rangle| \leq 2\sqrt{2}. \quad (4.2)$$

Here  $\langle A_x B_y C_z \rangle = \sum_{abc} a b c P(a, b, c | x, y, z)$ . Here it may be noted that the violation of the inequality initially proposed by Mermin [137] (which is nothing but  $M \leq 2$ ) in general, does not detect genuine entanglement. Subsequently, the above inequality (4.2) has been derived in order to detect genuine multipartite entanglement [130, 147]. Since quantum violation of the above inequality (4.2) can be detected by observing the outcome statistics of the local measurements alone, it enables detecting genuine entanglement without considering the dimension of the corresponding Hilbert space, and is hence, device independent.

With the motivation of getting stronger device-independent genuine entanglement witness, Uffink designed another nonlinear Bell-type inequality [132] which may distinguish genuine multipartite entanglement from lesser entangled states:

$$U = \langle A_1 B_0 C_0 + A_0 B_1 C_0 + A_0 B_0 C_1 - A_1 B_1 C_1 \rangle^2 + \langle A_1 B_1 C_0 + A_0 B_1 C_1 + A_1 B_0 C_1 - A_0 B_0 C_0 \rangle^2 \leq 8. \quad (4.3)$$

So far we have discussed the detection of genuine entanglement by looking at the measurement statistics in a device-independent way. However, there exist scenarios in which the devices are trusted, and one need not resort to the more resource consuming method of device-independent entanglement verification. We now describe the concept of witness operators which can also be used to detect genuine entanglement. A witness operator  $\mathcal{W}$  which detects genuine entanglement of a state  $\rho$  is a Hermitian operator that satisfies the conditions,

$$\begin{aligned} \text{Tr}(\mathcal{W}\rho) &\geq 0, \quad \forall \rho \in \mathcal{BS} \\ \exists \text{ at least one } \rho &\notin \mathcal{BS}, \quad \text{s.t. } \text{Tr}(\mathcal{W}\rho) < 0 \end{aligned} \quad (4.4)$$

where  $\mathcal{BS}$  is the set of all bi-separable states. The existence of such a witness operator is a consequence of the Hahn-Banach theorem on normed linear spaces [143]. For every genuinely entangled state, there exists a genuine entanglement witness.

In the present study we consider two types of witness operators that detect genuine entangled states. The first witness operator that we will use is suitable for detecting genuine entanglement of the three-qubit  $W$ -state. Consider the three-qubit  $W$  state given by,  $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ . The witness operator that detects genuine entanglement in the state  $|W\rangle$  is given by [53, 144–146],

$$\mathcal{W}_W = \frac{2}{3}\mathbb{I}_3 - |W\rangle\langle W|. \quad (4.5)$$

Whenever a state  $\rho$  gives  $\text{Tr}[\mathcal{W}_W\rho] < 0$ , genuine entanglement in the state  $\rho$  is certified.

Next we discuss the witness operator which is suitable for detecting genuine entanglement of three-qubit GHZ-state. Consider the three-qubit GHZ state given by,  $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ . The witness operator that detects genuine entanglement in the state  $|GHZ\rangle$  is given by [53, 144]

$$\mathcal{W}_{GHZ} = \frac{1}{2}\mathbb{I}_3 - |GHZ\rangle\langle GHZ|. \quad (4.6)$$

If a state  $\rho$  gives  $\text{Tr}[\mathcal{W}_{GHZ}\rho] < 0$ , then genuine entanglement in the state  $\rho$  is certified.

The advantage of such kind of witness operators is that they can be implemented in the laboratory by performing a finite number of correlated local measurements. Hence, such witness operators can be realized when the observers sharing the quantum state are spatially separated. The witness operator (4.5) can be written in the following decomposition into a sum of tensor products of operators:

$$\begin{aligned}
\mathcal{W}_W = & \\
& \frac{1}{24} \left( 13\mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} + 3\sigma_z \otimes \mathbb{I} \otimes \mathbb{I} + 3\mathbb{I} \otimes \sigma_z \otimes \mathbb{I} + 3\mathbb{I} \otimes \mathbb{I} \otimes \sigma_z + 5\sigma_z \otimes \sigma_z \otimes \mathbb{I} + 5\sigma_z \otimes \mathbb{I} \otimes \sigma_z \right. \\
& + 5\mathbb{I} \otimes \sigma_z \otimes \sigma_z + 7\sigma_z \otimes \sigma_z \otimes \sigma_z - \mathbb{I} \otimes \mathbb{I} \otimes (\sigma_z + \sigma_x) - \mathbb{I} \otimes (\sigma_z + \sigma_x) \otimes \mathbb{I} - (\sigma_z + \sigma_x) \otimes \mathbb{I} \\
& \otimes \mathbb{I} - \mathbb{I} \otimes (\sigma_z + \sigma_x) \otimes (\sigma_z + \sigma_x) - (\sigma_z + \sigma_x) \otimes \mathbb{I} \otimes (\sigma_z + \sigma_x) - (\sigma_z + \sigma_x) \otimes (\sigma_z + \sigma_x) \otimes \mathbb{I} \\
& - (\sigma_z + \sigma_x) \otimes (\sigma_z + \sigma_x) \otimes (\sigma_z + \sigma_x) - \mathbb{I} \otimes \mathbb{I} \otimes (\sigma_z - \sigma_x) - \mathbb{I} \otimes (\sigma_z - \sigma_x) \otimes \mathbb{I} - (\sigma_z - \sigma_x) \\
& \otimes \mathbb{I} \otimes \mathbb{I} - \mathbb{I} \otimes (\sigma_z - \sigma_x) \otimes (\sigma_z - \sigma_x) - (\sigma_z - \sigma_x) \otimes \mathbb{I} \otimes (\sigma_z - \sigma_x) - (\sigma_z - \sigma_x) \otimes (\sigma_z - \sigma_x) \\
& \otimes \mathbb{I} - (\sigma_z - \sigma_x) \otimes (\sigma_z - \sigma_x) \otimes (\sigma_z - \sigma_x) - \mathbb{I} \otimes \mathbb{I} \otimes (\sigma_z + \sigma_y) - \mathbb{I} \otimes (\sigma_z + \sigma_y) \otimes \mathbb{I} - (\sigma_z + \sigma_y) \\
& \otimes \mathbb{I} \otimes \mathbb{I} - \mathbb{I} \otimes (\sigma_z + \sigma_y) \otimes (\sigma_z + \sigma_y) - (\sigma_z + \sigma_y) \otimes \mathbb{I} \otimes (\sigma_z + \sigma_y) - (\sigma_z + \sigma_y) \otimes (\sigma_z + \sigma_y) \otimes \mathbb{I} \\
& - (\sigma_z + \sigma_y) \otimes (\sigma_z + \sigma_y) \otimes (\sigma_z + \sigma_y) - \mathbb{I} \otimes \mathbb{I} \otimes (\sigma_z - \sigma_y) - \mathbb{I} \otimes (\sigma_z - \sigma_y) \otimes \mathbb{I} - (\sigma_z - \sigma_y) \otimes \mathbb{I} \\
& \otimes \mathbb{I} - \mathbb{I} \otimes (\sigma_z - \sigma_y) \otimes (\sigma_z - \sigma_y) - (\sigma_z - \sigma_y) \otimes \mathbb{I} \otimes (\sigma_z - \sigma_y) - (\sigma_z - \sigma_y) \otimes (\sigma_z - \sigma_y) \otimes \mathbb{I} \\
& \left. - (\sigma_z - \sigma_y) \otimes (\sigma_z - \sigma_y) \otimes (\sigma_z - \sigma_y) \right). \tag{4.7}
\end{aligned}$$

Note that all the correlations of measurements like  $\sigma_z \otimes \sigma_z \otimes \sigma_z$ ,  $\sigma_z \otimes \sigma_z \otimes \mathbb{I}$ ,  $\sigma_z \otimes \mathbb{I} \otimes \sigma_z$ ,  $\mathbb{I} \otimes \sigma_z \otimes \sigma_z$ ,  $\sigma_z \otimes \mathbb{I} \otimes \mathbb{I}$ ,  $\mathbb{I} \otimes \sigma_z \otimes \mathbb{I}$ ,  $\mathbb{I} \otimes \mathbb{I} \otimes \sigma_z$  can be determined from the same data. Hence, the above decomposition requires measurements of five correlations:

- $\sigma_z \otimes \sigma_z \otimes \sigma_z$ ,
- $\left( \frac{\sigma_z + \sigma_x}{\sqrt{2}} \right) \otimes \left( \frac{\sigma_z + \sigma_x}{\sqrt{2}} \right) \otimes \left( \frac{\sigma_z + \sigma_x}{\sqrt{2}} \right)$ ,
- $\left( \frac{\sigma_z - \sigma_x}{\sqrt{2}} \right) \otimes \left( \frac{\sigma_z - \sigma_x}{\sqrt{2}} \right) \otimes \left( \frac{\sigma_z - \sigma_x}{\sqrt{2}} \right)$ ,
- $\left( \frac{\sigma_z + \sigma_y}{\sqrt{2}} \right) \otimes \left( \frac{\sigma_z + \sigma_y}{\sqrt{2}} \right) \otimes \left( \frac{\sigma_z + \sigma_y}{\sqrt{2}} \right)$ ,
- $\left( \frac{\sigma_z - \sigma_y}{\sqrt{2}} \right) \otimes \left( \frac{\sigma_z - \sigma_y}{\sqrt{2}} \right) \otimes \left( \frac{\sigma_z - \sigma_y}{\sqrt{2}} \right)$ .

Similarly, the witness operator (4.6) can be written in the following decomposition:

$$\begin{aligned}
\mathcal{W}_{GHZ} = & \frac{1}{8} \left( 3\mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} - \mathbb{I} \otimes \sigma_z \otimes \sigma_z - \sigma_z \otimes \mathbb{I} \otimes \sigma_z - \sigma_z \otimes \sigma_z \otimes \mathbb{I} - 2\sigma_x \otimes \sigma_x \otimes \sigma_x \right. \\
& \left. + \frac{1}{2}(\sigma_x + \sigma_y) \otimes (\sigma_x + \sigma_y) \otimes (\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y) \otimes (\sigma_x - \sigma_y) \otimes (\sigma_x - \sigma_y) \right). \tag{4.8}
\end{aligned}$$

The above decomposition requires measurements of four correlations:

- $\sigma_z \otimes \sigma_z \otimes \sigma_z$ ,
- $\sigma_x \otimes \sigma_x \otimes \sigma_x$ ,
- $\left( \frac{\sigma_x + \sigma_y}{\sqrt{2}} \right) \otimes \left( \frac{\sigma_x + \sigma_y}{\sqrt{2}} \right) \otimes \left( \frac{\sigma_x + \sigma_y}{\sqrt{2}} \right)$ ,

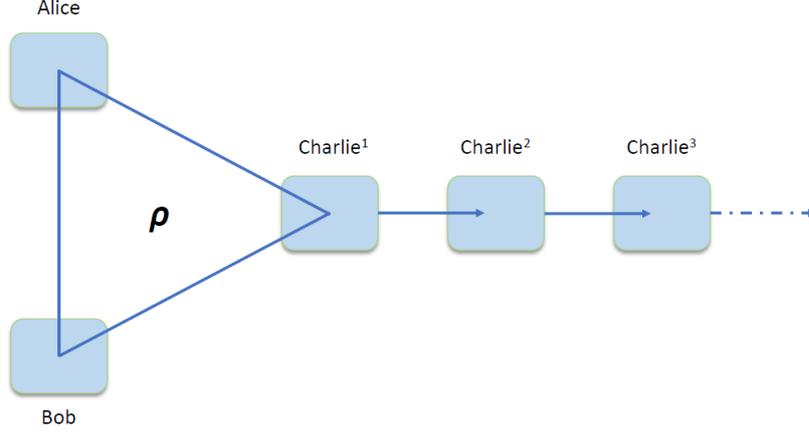


Figure 4.1: Sequential detection of genuine tripartite entanglement: consider a scenario where three spin- $\frac{1}{2}$  particles are prepared in the state  $\rho$ . Initially  $\rho$  is shared between three spatially separated parties say, Alice, Bob and Charlie<sup>1</sup>. Alice as well as Bob perform projective measurements on their respective particles but Charlie<sup>1</sup> performs unsharp measurements and sends the particle to Charlie<sup>2</sup>. After doing a measurement on his respective part Charlie<sup>2</sup> again sends the particle to Charlie<sup>3</sup>. In this way the protocol goes on.

$$\bullet \left( \frac{\sigma_x - \sigma_y}{\sqrt{2}} \right) \otimes \left( \frac{\sigma_x - \sigma_y}{\sqrt{2}} \right) \otimes \left( \frac{\sigma_x - \sigma_y}{\sqrt{2}} \right).$$

#### 4.2.2 Setting up the measurement context

In this subsection we describe the scenario adopted in the present study. Let us consider that three spatially separated observers say Alice, Bob and a sequence of multiple Charlies (i.e., Charlie<sup>1</sup>, Charlie<sup>2</sup>, Charlie<sup>3</sup>, ..., Charlie<sup>n</sup>) share a tripartite state  $\rho$  consisting of three spin- $\frac{1}{2}$  particles. In our scenario, Alice performs projective measurements on the first particle, Bob performs projective measurements on the second particle and multiple Charlies are allowed to perform non-projective or unsharp measurements [114, 115] on the third particle sequentially. Let us now clarify the measurement scenario of multiple Charlies. Initially, Charlie<sup>1</sup> performs an unsharp measurement on the third particle, then he sends that particle to Charlie<sup>2</sup>. Charlie<sup>2</sup> subsequently passes the third particle to Charlie<sup>3</sup> after performing another unsharp measurement. Charlie<sup>3</sup> also follows the same procedure and so on. This scenario is depicted in Fig. 4.1.

It may be noted here that the choice of measurement settings of each Charlie is independent and uncorrelated with the choices of measurement settings and outcomes of the

previous Charlies. The unbiased input scenario is another assumption that we have adopted in this study. It implies that all possible measurement settings of each Charlie are equally probable. Note also, that the no-signaling condition (the probability of obtaining one party's outcome does not depend on the other spatially separated party's setting) is satisfied between Alice, Bob and any Charlie as they are spatially separated and they perform measurements on three different particles. However, the no-signaling condition is not satisfied between different Charlies as each subsequent Charlie perform measurements on the same particle accessed earlier by the previous Charlie.

In the above scenario, we ask the question as to how many Charlies can detect genuine tripartite entanglement with Alice and Bob. We will address this issue by investigating how many Charlies can have correlations with Alice and Bob such that they violate the Mermin inequality (4.2) or the Uffink inequality (4.3). Furthermore, we will also discuss how many Charlies can demonstrate genuine tripartite entanglement if they use the witness operators given by Eq. (4.5) and Eq. (4.6), respectively. Here, if any Charlie performs projective measurements, then the entanglement of the state will be completely lost, and there will be no chance to detect entanglement by the subsequent Charlies. However, it is natural that no such restriction is required for the measurements performed by the last Charlie in the sequence. Hence, in order to deal with the above problem with  $n$  Charlies, the first  $(n - 1)$  Charlies in the sequence should perform unsharp measurements. In the following, we will briefly discuss the unsharp measurement formalism used in this study (For details, see [112, 113, 118]).

Following the standard projective measurement scheme proposed by von Neumann [45], after an interaction with a meter having the state  $\phi(q)$  ( $q$  is the position of the pointer), the state  $|\psi\rangle = a|0\rangle + b|1\rangle$  ( $|0\rangle$  and  $|1\rangle$  form orthonormal basis in  $\mathbb{C}^2$ ,  $|a|^2 + |b|^2 = 1$ ) of the system (to be measured) of a spin- $\frac{1}{2}$  particle becomes

$$|\psi\rangle \otimes \phi(q) \rightarrow a|0\rangle \otimes \phi(q - 1) + b|1\rangle \otimes \phi(q + 1). \quad (4.9)$$

In a general sharp or projective measurement, one obtains the maximum amount of information at the cost of maximum disturbance to the state of the system. On the other hand, the disturbance to the state can be reduced by performing an unsharp measurement where one

obtains less amount of information. An unsharp measurement can be characterised by two real parameters: the quality factor  $F$  and the precision  $G$  of the measurements. The quality factor quantifies the extend to which the initial state of the system (to be measured) remains undisturbed during the measurement process. Mathematically, the quality factor is defined as  $F(\phi(q)) = \int_{-\infty}^{\infty} \langle \phi(q+1) | \phi(q-1) \rangle dq$ . Precision  $G$  quantifies the information gain due to the measurement. Mathematically, it is defined as  $G(\phi(q)) = \int_{-1}^1 \phi^2(q) dq$ . It is obvious that for sharp measurement  $F = 0$  and  $G = 1$ . An optimal pointer state is the one for which one obtains the greatest precision for a given quality factor. The information-disturbance trade-off relation for an optimal pointer is given by,  $F^2 + G^2 = 1$  [112]. In other words, for dichotomic measurements on a qubit system, satisfying the condition:  $F^2 + G^2 = 1$  implies that the disturbance is minimized given a certain information gain.

The above formalism can be recast in terms of unsharp measurements. Unsharp measurement is one particular class of POVMs [114, 115]. POVM is nothing but set of positive operators that add to identity, i. e.,  $E \equiv \{E_i | \sum_i E_i = \mathbb{I}, 0 < E_i \leq \mathbb{I}\}$ . Here, each of the operators  $E_i$  determines the probability  $\text{Tr}[\rho E_i]$  of obtaining the  $i^{\text{th}}$  outcome (here  $\rho$  is the state of the system on which the measurement is performed).

In order to see how the unsharp measurement formalism [114, 115, 148–152] is connected with the one-parameter class of POVMs, consider a dichotomic observable  $A = P_+ - P_-$  with outcomes  $+1$  and  $-1$ , where  $P_+$  ( $P_-$ ) denotes the projectors associated with the outcome  $+1$  ( $-1$ );  $P_+ + P_- = \mathbb{I}$  and  $P_{\pm}^2 = P_{\pm}$ . Given the observable  $A$ , one can define a dichotomic unsharp observable  $A^\lambda = E_+^\lambda - E_-^\lambda$  [151, 152] associated with the sharpness parameter  $\lambda \in (0, 1]$ , where  $E_+^\lambda + E_-^\lambda = \mathbb{I}$ ,  $0 < E_{\pm}^\lambda \leq 1$ . Here the positive operators  $E_{\pm}^\lambda$  (also known as effect operators) are given by,

$$E_{\pm}^\lambda = \lambda P_{\pm} + (1 - \lambda) \frac{\mathbb{I}_2}{2}. \quad (4.10)$$

This is obtained by mixing projective measurements with white noise. The probability of getting the outcomes  $+1$  and  $-1$ , when the above unsharp measurement is performed on the

state  $\rho$ , are given by  $\text{Tr}[\rho E_+^\lambda]$  and  $\text{Tr}[\rho E_-^\lambda]$  respectively. Note that the above positive operators can also be written in the following way,

$$E_\pm^\lambda = \frac{1+\lambda}{2}P_\pm + \frac{1-\lambda}{2}P_\mp.$$

The expectation value of  $A^\lambda$  for a given  $\rho$  is defined as [151, 152],

$$\begin{aligned} \langle A^\lambda \rangle &= \text{Tr}[\rho E_+^\lambda] - \text{Tr}[\rho E_-^\lambda] \\ &= \text{Tr}[\rho(E_+^\lambda - E_-^\lambda)] \\ &= \lambda \langle A \rangle, \end{aligned} \tag{4.11}$$

where  $\langle A \rangle$  is the expectation value of the observable  $A$  under projective measurements. The operational meaning of the expectation value  $\langle A^\lambda \rangle$  follows from the above equation: from the probabilities ( $\text{Tr}[\rho E_\pm^\lambda]$ ) of obtaining the outcomes  $\pm 1$  under unsharp measurement, one can evaluate  $\langle A^\lambda \rangle$ . Note that these probabilities under unsharp measurements can be realised in experiments [153–156].

Using the generalized von Neumann-Lüders transformation rule [114], the states after the measurements, when the outcomes  $+1$  and  $-1$  occurs, are given by,  $\frac{\sqrt{E_+^\lambda}\rho\sqrt{E_+^\lambda}}{\text{Tr}[E_+^\lambda\rho]}$  and  $\frac{\sqrt{E_-^\lambda}\rho\sqrt{E_-^\lambda}}{\text{Tr}[E_-^\lambda\rho]}$  respectively. In any sequential measurement scenario, we need to gain certain information while minimally disturbing the state of the system. In case of qubits, unsharp measurements are shown to be good choice for this purpose [112]. For the von Neumann-Lüders transformation rule, it was shown [113] that the quality factor and the precision associated with the above unsharp measurement formalism are given by,  $F = \sqrt{1-\lambda^2}$  and  $G = \lambda$ . Hence, the optimality condition for information gain and disturbance,  $F^2 + G^2 = 1$  for qubits is compatible with the unsharp measurement formalism [113, 118]. In other words, the unsharp measurement formalism along with the von Neumann-Lüders transformation rule provides the largest amount of information for a given amount of disturbance created on the state due to the measurement.

In our study we will consider that each Charlie, except the final Charlie in the sequence, performs unsharp measurements.

### 4.3 SEQUENTIAL DETECTION OF GENUINE TRIPARTITE ENTANGLEMENT IN THE DEVICE-INDEPENDENT SCENARIO

In this section we find out the maximum number of Charlies that can independently and sequentially detect genuine entanglement in device-independent scenario. Before proceeding, we wish to mention that it was shown [129] earlier that at most two Charlies can simultaneously demonstrate genuine tripartite nonlocality with a single Alice and a single Bob in the scenario described in subsection 4.2.2. This was demonstrated through the quantum violation of the Svetlichny inequality [157]. Since genuine entanglement is necessary for demonstrating genuine nonlocality, we can state that at least two Charlies can simultaneously demonstrate genuine tripartite entanglement in a device-independent way with a single Alice and a single Bob through the quantum violation of the Svetlichny inequality [157]. Next, we want to find out whether the number of Charlies, who can detect genuine entanglement sequentially, can be increased using quantum violations of Mermin inequality (4.2) or Uffink inequality (4.3) in the scenario described in subsection 4.2.2.

We start with the Mermin inequality (4.2), which is maximally violated by tripartite GHZ state  $\rho_{GHZ} = |\psi_{GHZ}\rangle\langle\psi_{GHZ}|$ , where

$$|\psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle). \quad (4.12)$$

Suppose a tripartite GHZ state given by Eq.(4.12) is initially shared among Alice, Bob and multiple Charlies. Alice performs dichotomic sharp measurement of spin component observable on her part in the direction  $\hat{x}_0$ , or  $\hat{x}_1$ . Bob performs dichotomic sharp measurement of spin component observable on his particle in the direction  $\hat{y}_0$  or  $\hat{y}_1$ . Charlie<sup>m</sup> (where  $m \in \{1, 2, \dots, n\}$ ) performs dichotomic unsharp measurement of spin component observable in the direction  $\hat{z}_0^m$  or  $\hat{z}_1^m$ . The outcomes of each measurement are  $\pm 1$ .

The projectors associated with Alice's sharp measurement of spin component observable in the direction  $\hat{x}_i$  (with  $i \in \{0, 1\}$ ) can be written as  $P_{a|\hat{x}_i} = \frac{\mathbb{I}_2 + a \hat{x}_i \cdot \vec{\sigma}}{2}$  (with  $a$  being the outcome of the sharp measurement and  $a \in \{+1, -1\}$ ). Similarly, the projectors associated with Bob's sharp measurement of spin component observable in the direction  $\hat{y}_j$  (with  $j \in$

$\{0, 1\}$  are given by,  $P_{b|\hat{y}_j} = \frac{\mathbb{I}_2 + b \hat{y}_j \cdot \vec{\sigma}}{2}$  (with  $b$  being the outcome of the sharp measurement and  $b \in \{+1, -1\}$ ). The directions  $\hat{x}_i$  and  $\hat{y}_j$  can be expressed as,

$$\hat{\xi}_l = \sin \theta_l^\xi \cos \phi_l^\xi \hat{X} + \sin \theta_l^\xi \sin \phi_l^\xi \hat{Y} + \cos \theta_l^\xi \hat{Z}, \quad (4.13)$$

where  $l \in \{0, 1\}$ ;  $0 \leq \theta_l^\xi \leq \pi$ ;  $0 \leq \phi_l^\xi \leq 2\pi$ .  $\hat{X}, \hat{Y}, \hat{Z}$  are three orthogonal unit vectors in Cartesian coordinates. For Alice  $\xi = x$  and for Bob  $\xi = y$ .

The effect operators associated with Charlie<sup>*m*</sup>'s ( $m \in \{1, 2, \dots, n\}$ ) unsharp measurement of spin component observable in the direction  $\hat{z}_k^m$  (with  $k \in \{0, 1\}$ ) are given by,

$$E_{c^m|\hat{z}_k^m}^{\lambda_m} = \lambda_m \frac{\mathbb{I}_2 + c^m \hat{z}_k^m \cdot \vec{\sigma}}{2} + (1 - \lambda_m) \frac{\mathbb{I}_2}{2}, \quad (4.14)$$

where  $c^m$  is the outcome of the unsharp measurement by Charlie<sup>*m*</sup> and  $c^m \in \{+1, -1\}$ ;  $\lambda_m$  (with  $0 < \lambda_m \leq 1$ ) denotes the sharpness parameter associated with Charlie<sup>*m*</sup>'s unsharp measurement. When we consider a sequence of  $n$  Charlies, then the measurements of Charlie<sup>*n*</sup> will be sharp, i.e.,  $\lambda_n = 1$ . The direction  $\hat{z}_k^m$  is expressed as

$$\hat{z}_k^m = \sin \theta_k^{z^m} \cos \phi_k^{z^m} \hat{X} + \sin \theta_k^{z^m} \sin \phi_k^{z^m} \hat{Y} + \cos \theta_k^{z^m} \hat{Z}, \quad (4.15)$$

where  $0 \leq \theta_k^{z^m} \leq \pi$ ;  $0 \leq \phi_k^{z^m} \leq 2\pi$ .

The joint probability distribution of occurrence of the outcomes  $a, b, c^1$ , when Alice, Bob perform projective measurements of spin component observables along the directions  $\hat{x}_i$  and  $\hat{y}_j$  respectively, and Charlie<sup>1</sup> performs unsharp measurement of spin component observable along the direction  $\hat{z}_k^1$ , is given by,

$$P(a, b, c^1 | \hat{x}_i, \hat{y}_j, \hat{z}_k^1) = \text{Tr} \left[ \left\{ \frac{\mathbb{I}_2 + a \hat{x}_i \cdot \vec{\sigma}}{2} \otimes \frac{\mathbb{I}_2 + b \hat{y}_j \cdot \vec{\sigma}}{2} \otimes E_{c^1|\hat{z}_k^1}^{\lambda_1} \right\} \cdot \rho_{GHZ} \right]. \quad (4.16)$$

The correlation function between Alice, Bob and Charlie<sup>1</sup>, when Alice, Bob perform projective measurements of spin component observables along the directions  $\hat{x}_i$  and  $\hat{y}_j$  respectively

and Charlie<sup>1</sup> performs unsharp measurement of spin component observable along the direction  $\hat{z}_k^1$ , can be written as

$$C_{i,j,k}^1 = \sum_{a=-1}^{+1} \sum_{b=-1}^{+1} \sum_{c^1=-1}^{+1} a b c^1 P(a, b, c^1 | \hat{x}_i, \hat{y}_j, \hat{z}_k^1). \quad (4.17)$$

The left hand side of the Mermin inequality (4.2) associated with Alice, Bob and Charlie<sup>1</sup> in terms of the correlation functions is expressed as

$$M_1 = |C_{100}^1 + C_{010}^1 + C_{001}^1 - C_{111}^1|. \quad (4.18)$$

Now, it is observed that Alice, Bob and Charlie<sup>1</sup> get quantum violation of Mermin inequality (4.2) (i.e.,  $M_1 > 2\sqrt{2}$ ) when  $\lambda_1 > \frac{1}{\sqrt{2}}$ . This happens for the following choice of measurement settings:  $(\theta_0^x, \phi_0^x, \theta_1^x, \phi_1^x, \theta_0^y, \phi_0^y, \theta_1^y, \phi_1^y, \theta_0^{z^1}, \phi_0^{z^1}, \theta_1^{z^1}, \phi_1^{z^1}) \equiv (\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0)$ .

Charlie<sup>1</sup> passes his particle to Charlie<sup>2</sup> after his measurement. The following expression gives the unnormalized post measurement reduced state at Charlie<sup>2</sup>'s end after Alice, Bob get outcomes  $a, b$  by performing projective measurements of spin component observables along the directions  $\hat{x}_i$  and  $\hat{y}_j$  respectively and Charlie<sup>1</sup> gets outcome  $c^1$  by performing unsharp measurement of spin component observable along the direction  $\hat{z}_k^1$ :

$$\rho_{un}^{c^2} = \text{Tr}_{AB} \left[ \left\{ \frac{\mathbb{I}_2 + a\hat{x}_i \cdot \vec{\sigma}}{2} \otimes \frac{\mathbb{I}_2 + b\hat{y}_j \cdot \vec{\sigma}}{2} \otimes \sqrt{E_{c^1|\hat{z}_k^1}^{\lambda_1}} \right\} \cdot \rho_{GHZ} \cdot \left\{ \frac{\mathbb{I}_2 + a\hat{x}_i \cdot \vec{\sigma}}{2} \otimes \frac{\mathbb{I}_2 + b\hat{y}_j \cdot \vec{\sigma}}{2} \otimes \sqrt{E_{c^1|\hat{z}_k^1}^{\lambda_1}} \right\} \right], \quad (4.19)$$

where,

$$\sqrt{E_{c^1|\hat{z}_k^1}^{\lambda_1}} = \sqrt{\frac{1 + \lambda_1}{2}} \left( \frac{\mathbb{I}_2 + c^1 \hat{z}_k^1 \cdot \vec{\sigma}}{2} \right) + \sqrt{\frac{1 - \lambda_1}{2}} \left( \frac{\mathbb{I}_2 - c^1 \hat{z}_k^1 \cdot \vec{\sigma}}{2} \right). \quad (4.20)$$

Here  $\text{Tr}_{AB}[\dots]$  denotes partial trace over the subsystems of Alice and Bob. On the above reduced state, Charlie<sup>2</sup> again performs unsharp measurement (with sharpness parameter being denoted by  $\lambda_2$ ) of spin component observable along the direction  $\hat{z}_l^2$  and gets the outcome  $c^2$ . The joint probability distribution of occurrence of the outcomes  $a, b, c^1, c^2$ , when Alice, Bob perform projective measurements of spin component observables along the direc-

tions  $\hat{x}_i$  and  $\hat{y}_j$  respectively and Charlie<sup>1</sup>, Charlie<sup>2</sup> perform unsharp measurement of spin component observable along the direction  $\hat{z}_k^1, \hat{z}_l^2$  respectively, is given by,

$$P(a, b, c^1, c^2 | \hat{x}_i, \hat{y}_j, \hat{z}_k^1, \hat{z}_l^2) = \text{Tr} \left[ E_{c^2 | \hat{z}_l^2}^{\lambda_2} \cdot \rho_{un}^{C^2} \right]. \quad (4.21)$$

From this expression, one can obtain the joint probability of obtaining the outcomes  $a, b, c^2$  when Alice, Bob, Charlie<sup>2</sup> measures spin component observables in the directions  $\hat{x}_i, \hat{y}_j, \hat{z}_l^2$ , respectively and when Charlie<sup>1</sup> has already measured spin component observables in the directions  $\hat{z}_k^1$ :

$$P(a, b, c^2 | \hat{x}_i, \hat{y}_j, \hat{z}_k^1, \hat{z}_l^2) = \sum_{c^1=-1}^{+1} P(a, b, c^1, c^2 | \hat{x}_i, \hat{y}_j, \hat{z}_k^1, \hat{z}_l^2). \quad (4.22)$$

Let  $C_{ijkl}^2$  denote the correlation between Alice, Bob and Charlie<sup>2</sup> when Alice, Bob, Charlie<sup>1</sup> and Charlie<sup>2</sup> measure spin component observables in the directions  $\hat{x}_i, \hat{y}_j, \hat{z}_k^1$  and  $\hat{z}_l^2$ , respectively. The expression for  $C_{ijkl}^2$  can be obtained from

$$C_{ijkl}^2 = \sum_{a=-1}^{+1} \sum_{b=-1}^{+1} \sum_{c^2=-1}^{+1} a b c^2 P(a, b, c^2 | \hat{x}_i, \hat{y}_j, \hat{z}_k^1, \hat{z}_l^2). \quad (4.23)$$

Since Charlie<sup>2</sup>'s choice of measurement settings is independent of the measurement settings of Charlie<sup>1</sup>, the above correlation has to be averaged over the two possible measurement settings of Charlie<sup>1</sup> (spin component observables in the directions  $\{\hat{z}_0^1, \hat{z}_1^1\}$ ). This average correlation function between Alice, Bob and Charlie<sup>2</sup> is given by,

$$\overline{C_{ijl}^2} = \sum_{k=0,1} C_{ijkl}^2 P(\hat{z}_k^1), \quad (4.24)$$

where  $P(\hat{z}_k^1)$  is the probability with which Charlie<sup>1</sup> performs unsharp measurement of spin component observables in the direction  $\hat{z}_k^1$  ( $k \in \{0, 1\}$ ). For an unbiased input scenario, we take the two measurement settings for Charlie<sup>1</sup> to be equally probable, i.e.,  $P(\hat{z}_0^1) = P(\hat{z}_1^1) = \frac{1}{2}$ .

The left hand side of the Mermin inequality (4.2) associated with Alice, Bob and Charlie<sup>2</sup> in terms of the average correlation functions is expressed as

$$M_2 = |\overline{C_{100}^2} + \overline{C_{010}^2} + \overline{C_{001}^2} - \overline{C_{111}^2}|. \quad (4.25)$$

In a similar way by evaluating the average correlation functions between Alice, Bob and Charlie<sup>m</sup>, the Mermin inequality can be written as

$$M_m = |\overline{C_{100}^m} + \overline{C_{010}^m} + \overline{C_{001}^m} - \overline{C_{111}^m}| \leq 2\sqrt{2}. \quad (4.26)$$

Violation of this inequality implies detection of genuine entanglement by Alice, Bob and Charlie<sup>m</sup>.

Let us first study whether Charlie<sup>1</sup> and Charlie<sup>2</sup> can sequentially detect genuine entanglement through quantum violation of Mermin inequality (4.2) with a single Alice and a single Bob in the scenario depicted in Fig. 4.1. Since there are only two Charlies in this case, we consider measurements of Charlie<sup>2</sup> to be sharp, i.e.,  $\lambda_2 = 1$ . For the following measurement settings :  $(\theta_0^x, \phi_0^x, \theta_1^x, \phi_1^x, \theta_0^y, \phi_0^y, \theta_1^y, \phi_1^y, \theta_0^z, \phi_0^z, \theta_1^z, \phi_1^z, \theta_0^z, \phi_0^z, \theta_1^z, \phi_1^z) \equiv (\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0)$  and for  $\lambda_1 = 0.74$ , we observe that Charlie<sup>1</sup> gets 5% violation of the Mermin inequality (4.2) (i.e.,  $M_1 = 2.96$ ) and Charlie<sup>2</sup> gets 18% violation of the Mermin inequality (4.2) (i.e.,  $M_2 = 3.34$ ). Hence, Charlie<sup>1</sup> and Charlie<sup>2</sup> can detect genuine entanglement sequentially through the quantum violations of the Mermin inequality (4.2). In fact, it can be shown that Charlie<sup>1</sup> and Charlie<sup>2</sup> both get quantum violations of the Mermin inequality (4.2) when  $\lambda_1 \in (0.71, 0.91)$ .

Next, we investigate whether Charlie<sup>1</sup>, Charlie<sup>2</sup> and Charlie<sup>3</sup> can sequentially detect genuine entanglement through quantum violation of Mermin inequality (4.2) with single Alice and single Bob in the scenario depicted in Fig. 4.1. In this case the measurements of Charlie<sup>3</sup> will be sharp, i.e.,  $\lambda_3 = 1$ . On the other hand, Charlie<sup>1</sup> and Charlie<sup>2</sup> perform unsharp measurements. When Charlie<sup>1</sup> gets 5% violation and Charlie<sup>2</sup> gets 5% violation of the Mermin inequality (4.2) (i.e., when  $M_1 = 2.96$  and  $M_2 = 2.96$ ), then the maximum magnitude of left hand side of Mermin inequality (4.2) for Charlie<sup>3</sup> becomes  $M_3 = 2.62$ . This happens for the following choice of measurement settings:  $(\theta_0^x, \phi_0^x, \theta_1^x, \phi_1^x, \theta_0^y, \phi_0^y, \theta_1^y, \phi_1^y, \theta_0^z, \phi_0^z, \theta_1^z, \phi_1^z, \theta_0^z, \phi_0^z, \theta_1^z, \phi_1^z, \theta_0^z, \phi_0^z, \theta_1^z, \phi_1^z)$

$\phi_0^{z^2}, \theta_1^{z^2}, \phi_1^{z^2}, \theta_0^{z^3}, \phi_0^{z^3}, \theta_1^{z^3}, \phi_1^{z^3}) \equiv (\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0)$  and when  $\lambda_1 = 0.74$  and  $\lambda_2 = 0.88$ . In fact, when  $M_1 = 2\sqrt{2}$ ,  $M_2 = 2\sqrt{2}$ , then the maximum of  $M_3 = 2.78 < 2\sqrt{2}$ . Hence, Charlie<sup>1</sup>, Charlie<sup>2</sup>, Charlie<sup>3</sup> cannot detect genuine entanglement sequentially through the quantum violations of the Mermin inequality (4.2).

One important point to be noted here is that Charlie<sup>3</sup> may obtain a quantum violation of the Mermin inequality (4.2) if the sharpness parameter of Charlie<sup>2</sup> or that of Charlie<sup>1</sup> is too small to get a violation. Hence, at most two Charlies can sequentially detect genuine entanglement through quantum violations of the Mermin inequality (4.2).

Up to now we have used quantum violation of the Mermin inequality (4.2) to certify genuine entanglement between Alice, Bob and any Charlie. Now, we will investigate whether the number of Charlies who can sequentially detect genuine entanglement, can be increased by using quantum violation of the Uffink inequality (4.3). The Uffink inequality in terms of the average correlation functions between Alice, Bob and Charlie<sup>m</sup> can be expressed as

$$U_m = \left( \overline{C}_{100}^m + \overline{C}_{010}^m + \overline{C}_{001}^m - \overline{C}_{111}^m \right)^2 \left( \overline{C}_{110}^m + \overline{C}_{011}^m + \overline{C}_{101}^m - \overline{C}_{000}^m \right)^2 \leq 8. \quad (4.27)$$

The average correlation functions can be evaluated following the aforementioned procedure. Violation of this inequality implies that a genuine entangled state is shared between Alice, Bob and Charlie<sup>m</sup>. In this case too, we assume that the three qubit GHZ state is initially shared between Alice, Bob and Charlie<sup>1</sup> as this state can give the maximum quantum violation ( $U_m = 16$ ) of the Uffink inequality (4.3).

Let us try to find out whether Charlie<sup>1</sup>, Charlie<sup>2</sup> and Charlie<sup>3</sup> can sequentially detect genuine entanglement through quantum violation of Uffink inequality (4.3) with single Alice and single Bob. Here the measurements of Charlie<sup>3</sup> is sharp, i.e.,  $\lambda_3 = 1$ . When Charlie<sup>1</sup> gets 5% violation and Charlie<sup>2</sup> gets 5% violation of the Uffink inequality (4.3) (i.e., when  $U_1 = 8.40$  and  $U_2 = 8.40$ ), then the maximum magnitude of left hand side of Uffink inequality (4.3) for Charlie<sup>3</sup> becomes  $U_3 = 7.73$ . This happens for the following choice of measurement settings:  $(\theta_0^x, \phi_0^x, \theta_1^x, \phi_1^x, \theta_0^y, \phi_0^y, \theta_1^y, \phi_1^y, \theta_0^z, \phi_0^z, \theta_1^z, \phi_1^z, \theta_0^z, \phi_0^z, \theta_1^z, \phi_1^z, \theta_0^z, \phi_0^z, \theta_1^z, \phi_1^z) \equiv (\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, 0)$  and when  $\lambda_1 = 0.72$  and  $\lambda_2 = 0.86$ . In fact, we observe that when  $U_1 = 8$ ,  $U_2 = 8$ , then the maximum of  $U_3 = 7.76$ . Hence, at most

two Charlies can detect genuine entanglement sequentially through the quantum violations of the Uffink inequality (4.3).

#### 4.4 SEQUENTIAL DETECTION OF GENUINE TRIPARTITE ENTANGLEMENT USING WITNESS OPERATORS

In this section, in order to probe sequential detection of genuine entanglement by multiple Charlies in the scenario described in Section 4.2.2, we are going to use genuine entanglement witnesses, instead of using device-independent genuine entanglement inequalities. The first witness operator suitable for detecting genuine entanglement of the three qubit W-state is given by [53, 144–146],

$$\mathcal{W}_W = \frac{2}{3}\mathbb{I}_3 - |W\rangle\langle W|. \quad (4.28)$$

The decomposition of this witness operator in terms of tensor products of operators is given by Eq.(4.7). However, in the scenario depicted in Fig. 4.1 the local measurements performed by Charlie<sup>m</sup> (except for the final Charlie in a sequence) is unsharp. Since, the decomposition (4.7) of the witness operator  $\mathcal{W}_W$  can be used when each observer performs sharp projective measurements, we have to modify the decomposition (4.7) of the above witness operator for unsharp measurements at Charlie's end. In order to do this, we will follow the prescription described in [120].

The joint probability of obtaining the outcomes  $a, b, c^m$ , when Alice, Bob perform projective measurements of spin component observables along the directions  $\hat{x}_i$  and  $\hat{y}_j$  respectively and Charlie<sup>m</sup> performs unsharp measurement of spin component observable along the direction  $\hat{z}_k^m$ , can be evaluated using the formula,

$$\text{Tr} \left[ \rho \left( P_{a|\hat{x}_i} \otimes P_{b|\hat{y}_j} \otimes E_{c^m|\hat{z}_k^m}^{\lambda_m} \right) \right], \quad (4.29)$$

where  $\rho$  is the average post-measurement state obtained after the previous stage of the measurement processes;  $P_{a|\hat{x}_i}$  and  $P_{b|\hat{y}_j}$  are projection operators corresponding to the projective

measurements by Alice and Bob respectively, and  $E_{c^m|z_k^m}^{\lambda_m}$  is the effect operator associated with the POVM (unsharp measurement) performed by Charlie<sup>m</sup>.

The expectation value of the state  $\rho$  corresponding to the above joint measurements is given by,

$$\text{Tr} \left[ \left\{ \left( P_{+|\hat{x}_i} - P_{-|\hat{x}_i} \right) \otimes \left( P_{+|\hat{y}_j} - P_{-|\hat{y}_j} \right) \otimes \left( E_{+|z_k^m}^{\lambda_m} - E_{-|z_k^m}^{\lambda_m} \right) \right\} \rho \right]. \quad (4.30)$$

Now,  $P_{+|\hat{x}_i} - P_{-|\hat{x}_i}$  ( $P_{+|\hat{y}_j} - P_{-|\hat{y}_j}$ ) is nothing but  $\hat{x}_i \cdot \vec{\sigma}$  ( $\hat{y}_j \cdot \vec{\sigma}$ ). Let us denote it by  $\sigma_{x_i}$  ( $\sigma_{y_j}$ ). Let us also denote  $E_{+|z_k^m}^{\lambda_m} - E_{-|z_k^m}^{\lambda_m}$  as  $\sigma_{z_k^m}^{\lambda_m}$ . Hence, we can write the following,

$$\begin{aligned} & \langle \sigma_{x_i} \otimes \sigma_{y_j} \otimes \sigma_{z_k^m}^{\lambda_m} \rangle \\ &= \text{Tr} \left[ \left( P_{+|\hat{x}_i} - P_{-|\hat{x}_i} \right) \otimes \left( P_{+|\hat{y}_j} - P_{-|\hat{y}_j} \right) \otimes \left( E_{+|z_k^m}^{\lambda_m} - E_{-|z_k^m}^{\lambda_m} \right) \rho \right] \\ &= \text{Tr} \left[ \left( P_{+|\hat{x}_i} - P_{-|\hat{x}_i} \right) \otimes \left( P_{+|\hat{y}_j} - P_{-|\hat{y}_j} \right) \otimes \lambda_m \left( P_{+|z_k^m} - P_{-|z_k^m} \right) \rho \right] \\ &= \lambda_m \langle \sigma_{x_i} \otimes \sigma_{y_j} \otimes \sigma_{z_k^m} \rangle. \end{aligned} \quad (4.31)$$

Noting the above relation, one can use the substitution  $\langle \sigma_{x_i} \otimes \sigma_{y_j} \otimes \sigma_{z_k^m}^{\lambda_m} \rangle \rightarrow \lambda_m \langle \sigma_{x_i} \otimes \sigma_{y_j} \otimes \sigma_{z_k^m} \rangle$  in the case of a general  $\lambda_m$  [120]. The unsharp version decomposition (4.7) of the genuine entanglement witness operator  $\mathcal{W}_W^{\lambda_m}$  now takes the form,

$$\begin{aligned} \mathcal{W}_W^{\lambda_m} &= \frac{1}{24} \left( 13 \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} + 3 \sigma_z \otimes \mathbb{I} \otimes \mathbb{I} + 3 \mathbb{I} \otimes \sigma_z \otimes \mathbb{I} + 3 \mathbb{I} \otimes \mathbb{I} \otimes \lambda_m \sigma_z + 5 \sigma_z \otimes \sigma_z \otimes \mathbb{I} + 5 \sigma_z \right. \\ & \otimes \mathbb{I} \otimes \lambda_m \sigma_z + 5 \mathbb{I} \otimes \sigma_z \otimes \lambda_m \sigma_z + 7 \sigma_z \otimes \sigma_z \otimes \lambda_m \sigma_z - \mathbb{I} \otimes \mathbb{I} \otimes \lambda_m (\sigma_z + \sigma_x) - \mathbb{I} \otimes (\sigma_z + \sigma_x) \\ & \otimes \mathbb{I} - (\sigma_z + \sigma_x) \otimes \mathbb{I} \otimes \mathbb{I} - \mathbb{I} \otimes (\sigma_z + \sigma_x) \otimes \lambda_m (\sigma_z + \sigma_x) - (\sigma_z + \sigma_x) \otimes \mathbb{I} \otimes \lambda_m (\sigma_z + \sigma_x) - \\ & (\sigma_z + \sigma_x) \otimes (\sigma_z + \sigma_x) \otimes \mathbb{I} - (\sigma_z + \sigma_x) \otimes (\sigma_z + \sigma_x) \otimes \lambda_m (\sigma_z + \sigma_x) - \mathbb{I} \otimes \mathbb{I} \otimes \lambda_m (\sigma_z - \sigma_x) \\ & - \mathbb{I} \otimes (\sigma_z - \sigma_x) \otimes \mathbb{I} - (\sigma_z - \sigma_x) \otimes \mathbb{I} \otimes \mathbb{I} - \mathbb{I} \otimes (\sigma_z - \sigma_x) \otimes \lambda_m (\sigma_z - \sigma_x) - (\sigma_z - \sigma_x) \otimes \mathbb{I} \\ & \otimes \lambda_m (\sigma_z - \sigma_x) - (\sigma_z - \sigma_x) \otimes (\sigma_z - \sigma_x) \otimes \mathbb{I} - (\sigma_z - \sigma_x) \otimes (\sigma_z - \sigma_x) \otimes \lambda_m (\sigma_z - \sigma_x) - \mathbb{I} \\ & \otimes \mathbb{I} \otimes \lambda_m (\sigma_z + \sigma_y) - \mathbb{I} \otimes (\sigma_z + \sigma_y) \otimes \mathbb{I} - (\sigma_z + \sigma_y) \otimes \mathbb{I} \otimes \mathbb{I} - \mathbb{I} \otimes (\sigma_z + \sigma_y) \otimes \lambda_m (\sigma_z + \sigma_y) \\ & - (\sigma_z + \sigma_y) \otimes \mathbb{I} \otimes \lambda_m (\sigma_z + \sigma_y) - (\sigma_z + \sigma_y) \otimes (\sigma_z + \sigma_y) \otimes \mathbb{I} - (\sigma_z + \sigma_y) \otimes (\sigma_z + \sigma_y) \otimes \lambda_m \\ & (\sigma_z + \sigma_y) - \mathbb{I} \otimes \mathbb{I} \otimes \lambda_m (\sigma_z - \sigma_y) - \mathbb{I} \otimes (\sigma_z - \sigma_y) \otimes \mathbb{I} - (\sigma_z - \sigma_y) \otimes \mathbb{I} \otimes \mathbb{I} - \mathbb{I} \otimes (\sigma_z - \sigma_y) \otimes \\ & \lambda_m (\sigma_z - \sigma_y) - (\sigma_z - \sigma_y) \otimes \mathbb{I} \otimes \lambda_m (\sigma_z - \sigma_y) - (\sigma_z - \sigma_y) \otimes (\sigma_z - \sigma_y) \otimes \mathbb{I} - (\sigma_z - \sigma_y) \otimes (\sigma_z \\ & - \sigma_y) \otimes \lambda_m (\sigma_z - \sigma_y) \left. \right). \end{aligned} \quad (4.32)$$

Now, since we have  $\text{Tr}[\mathcal{W}_W \rho_{BS}] \geq 0 \forall \rho_{BS} \in \mathcal{BS}$  (where  $\mathcal{BS}$  is the set of all bi-seperable states) and  $0 < \lambda_m \leq 1$ , we can write the following:

$$\begin{aligned}
\text{Tr}[\mathcal{W}_W^{\lambda_m} \rho_{BS}] &= \lambda_m \text{Tr}[\mathcal{W}_W \rho_{BS}] \\
&+ \frac{1}{24}(1 - \lambda_m) \left( 13 + 3 \text{Tr}[\rho_{BS}(\sigma_z \otimes \mathbb{I} \otimes \mathbb{I})] + 3 \text{Tr}[\rho_{BS}(\mathbb{I} \otimes \sigma_z \otimes \mathbb{I})] + 5 \text{Tr}[\rho_{BS}(\sigma_z \otimes \sigma_z \otimes \mathbb{I})] \right. \\
&- \text{Tr}[\rho_{BS}(\mathbb{I} \otimes (\sigma_z + \sigma_x) \otimes \mathbb{I})] - \text{Tr}[\rho_{BS}((\sigma_z + \sigma_x) \otimes \mathbb{I} \otimes \mathbb{I})] - \text{Tr}[\rho_{BS}((\sigma_z + \sigma_x) \otimes (\sigma_z + \sigma_x) \otimes \mathbb{I})] \\
&- \text{Tr}[\rho_{BS}(\mathbb{I} \otimes (\sigma_z - \sigma_x) \otimes \mathbb{I})] - \text{Tr}[\rho_{BS}((\sigma_z - \sigma_x) \otimes \mathbb{I} \otimes \mathbb{I})] - \text{Tr}[\rho_{BS}((\sigma_z - \sigma_x) \otimes (\sigma_z - \sigma_x) \otimes \mathbb{I})] \\
&- \text{Tr}[\rho_{BS}(\mathbb{I} \otimes (\sigma_z + \sigma_y) \otimes \mathbb{I})] - \text{Tr}[\rho_{BS}((\sigma_z + \sigma_y) \otimes \mathbb{I} \otimes \mathbb{I})] - \text{Tr}[\rho_{BS}((\sigma_z + \sigma_y) \otimes (\sigma_z + \sigma_y) \otimes \mathbb{I})] \\
&- \left. \text{Tr}[\rho_{BS}((\sigma_z - \sigma_y) \otimes \mathbb{I} \otimes \mathbb{I})] - \text{Tr}[\rho_{BS}(\mathbb{I} \otimes (\sigma_z - \sigma_y) \otimes \mathbb{I})] - \text{Tr}[\rho_{BS}((\sigma_z - \sigma_y) \otimes (\sigma_z - \sigma_y) \otimes \mathbb{I})] \right) \\
&= \lambda_m \text{Tr}[\mathcal{W}_W \rho_{BS}] + \frac{1}{24}(1 - \lambda_m) \left( 13 - \langle \sigma_z \otimes \mathbb{I} \otimes \mathbb{I} \rangle - \langle \mathbb{I} \otimes \sigma_z \otimes \mathbb{I} \rangle + \langle \sigma_z \otimes \sigma_z \otimes \mathbb{I} \rangle - 2 \langle \sigma_x \otimes \sigma_x \otimes \mathbb{I} \rangle \right. \\
&\quad \left. - 2 \langle \sigma_y \otimes \sigma_y \otimes \mathbb{I} \rangle \right) \\
&\geq \lambda_m \text{Tr}[\mathcal{W}_W \rho_{BS}] + \frac{1}{4}(1 - \lambda_m) \geq 0 \quad \forall \rho_{BS} \in \mathcal{BS}. \tag{4.33}
\end{aligned}$$

The first inequality in the last line of (4.33) is obtained by minimizing all the expectation values. Hence, we can conclude that the operator  $\mathcal{W}_W$  even after introducing unsharpness in Charlie's measurements ( $\mathcal{W}_W^{\lambda_m}$ ) can be used as a valid witness of genuine entanglement.

Charlie <sup>m</sup>	Permissible ranges for $\lambda_m$
Charlie <sup>1</sup>	$1 \geq \lambda_1 > \lambda_1^{\min} = 0.54$
Charlie <sup>2</sup>	$1 \geq \lambda_2 > \lambda_2^{\min} = 0.60$ , when $\lambda_i = \lambda_i^{\min} \forall i < 2$
Charlie <sup>3</sup>	$1 \geq \lambda_3 > \lambda_3^{\min} = 0.69$ , when $\lambda_i = \lambda_i^{\min} \forall i < 3$
Charlie <sup>4</sup>	$1 \geq \lambda_4 > \lambda_4^{\min} = 0.84$ , when $\lambda_i = \lambda_i^{\min} \forall i < 4$
Charlie <sup>5</sup>	No valid permissible range for $\lambda_5$ , when $\lambda_i = \lambda_i^{\min} \forall i < 5$

Table 4.1: Here we show the permissible ranges of sharpness parameters  $\lambda_m$  (where  $0 < \lambda_m \leq 1$ ) of Charlie<sup>m</sup> for detecting genuine entanglement through the witness operator  $\mathcal{W}_W^{\lambda_m}$  with a single Alice and a single Bob at the other sides. The permissible range of each  $\lambda_m$  depends on the values  $\lambda_1, \lambda_2, \dots, \lambda_{m-1}$ . In the above table we have presented the permissible range of each  $\lambda_m$  for the minimum permissible values of  $\lambda_1, \lambda_2, \dots, \lambda_{m-1}$ . For other values of  $\lambda_1, \lambda_2, \dots, \lambda_{m-1}$ , the permissible range of each  $\lambda_m$  can also be calculated. However, the permissible ranges of  $\lambda_m$  will be smaller than that presented in the table if we take other values  $\lambda_i > \lambda_i^{\min} \forall i < m$ , and the maximum number of Charlies may get reduced. Here we find that at most four Charlies can detect genuine tripartite entanglement through the witness operator  $\mathcal{W}_W^{\lambda_m}$ .

Before proceeding further, let us elaborate on the scenario used by us in the context of genuine entanglement witness operators. As discussed earlier, we have considered that Alice, Bob and any Charlie are spatially separated from each other. The no-signaling condition is satisfied between Alice, Bob and any Charlie. Each of Alice, Bob and Charlie<sup>m</sup> (where  $m \geq 1$  is arbitrary) always performs any of the pre-defined (sharp or unsharp) measurements (e.g. in case of  $\mathcal{W}_W^{\lambda_m}$ ,  $\sigma_z$ ,  $(\sigma_z + \sigma_x)/\sqrt{2}$ ,  $(\sigma_z - \sigma_x)/\sqrt{2}$ ,  $(\sigma_z + \sigma_y)/\sqrt{2}$ ,  $(\sigma_z - \sigma_y)/\sqrt{2}$ ) randomly in any experimental run. After completion of the experiment, they communicate their choice of measurement setting and outcome for each of the experimental run to the referee. The referee then determines the correlations necessary to evaluate the witness. For example, when the referee wants to determine  $\langle \sigma_z \otimes \sigma_z \otimes \lambda_m \sigma_z \rangle$  between Alice, Bob and Charlie<sup>m</sup>, then the referee will only consider the data of those experimental runs in which each of Alice, Bob and Charlie<sup>m</sup> performs (sharp or unsharp) measurement of  $\sigma_z$ .

In case of  $\mathcal{W}_W^{\lambda_m}$ , Charlie<sup>m</sup> always measures unsharp version of any of the above-mentioned five observables. But the choice of the particular measurement (among the above five measurements) performed by Charlie<sup>m</sup> in a run is always independent of the particular measurements performed by previous Charlies. To summarize, any Charlie always performs one of the above-mentioned five measurements randomly and independently of the choices of measurements (among the above-mentioned five measurements) of the previous Charlies.

Now, for example, consider an experimental run in which Charlie<sup>1</sup> and Charlie<sup>2</sup> perform unsharp measurements of  $\sigma_z$  and  $(\sigma_z + \sigma_x)/\sqrt{2}$  respectively. Then Charlie<sup>1</sup>'s measurement outcome will be useful for the referee to calculate  $\langle \sigma_z \otimes \sigma_z \otimes \lambda_1 \sigma_z \rangle$  between Alice, Bob and Charlie<sup>1</sup> if Alice and Bob perform sharp measurements of  $\sigma_z$  in that experimental run. But Charlie<sup>2</sup>'s measurement outcomes will not be useful in this case to calculate any of the correlations necessary to evaluate the witness. On the other hand, Charlie<sup>2</sup>'s measurement outcome in the above experimental run will be useful for the referee to calculate  $\langle (\sigma_z + \sigma_x) \otimes (\sigma_z + \sigma_x) \otimes \lambda_2 (\sigma_z + \sigma_x) \rangle$  if Alice and Bob perform sharp measurements of  $(\sigma_z + \sigma_x)/\sqrt{2}$  in that experimental run. But Charlie<sup>1</sup>'s measurement outcomes will not be useful in this case.

Now, suppose that the three qubit W state given by,  $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$  is initially shared between Alice, Bob and Charlie<sup>1</sup>. When Alice, Bob perform projective mea-

measurements and Charlie<sup>1</sup> performs unsharp measurement with sharpness parameter being denoted by  $\lambda_1$ , the entanglement witness  $\mathcal{W}_W^{\lambda_1}$  acquires the following expectation value

$$\text{Tr} \left[ |W\rangle\langle W| \mathcal{W}_W^{\lambda_1} \right] = \frac{1}{18} (7 - 13\lambda_1). \quad (4.34)$$

It is clear from the above equation that Charlie<sup>1</sup> can detect genuine entanglement with Alice and Bob when  $\lambda_1 > \frac{7}{13} \simeq 0.54$ .

Let us now explore whether there is any possibility for subsequent Charlies, i.e, Charlie<sup>2</sup>, Charlie<sup>3</sup> ..., to detect the residual genuine entanglement in the post measurement average state with single Alice and single Bob at other sides. Since any Charlie is ignorant about the choices of measurement settings and outcomes all previous Charlies, we have to average over the previous Charlie's inputs and outputs to obtain the state shared between Alice, Bob and the Charlie of the current stage of the experiment. After performance of Charlie<sup>1</sup>'s unsharp measurement, the average state becomes,

$$|W\rangle\langle W| \rightarrow \rho_W^{\lambda_1} = \frac{1}{5} \sum_{i, \hat{z}_k^1} \left( \mathbb{I} \otimes \mathbb{I} \otimes \sqrt{E_{i|\hat{z}_k^1}^{\lambda_1}} \right) |W\rangle\langle W| \left( \mathbb{I} \otimes \mathbb{I} \otimes \sqrt{E_{i|\hat{z}_k^1}^{\lambda_1}} \right), \quad (4.35)$$

where  $i \in \{+1, -1\}$ ,  $\hat{z}_k^1 \in \left\{ \hat{z}, \frac{\hat{z} + \hat{x}}{\sqrt{2}}, \frac{\hat{z} - \hat{x}}{\sqrt{2}}, \frac{\hat{z} + \hat{y}}{\sqrt{2}}, \frac{\hat{z} - \hat{y}}{\sqrt{2}} \right\}$ .

In the next step Charlie<sup>2</sup> performs unsharp measurements on his part of  $\rho_W^{\lambda_1}$  with sharpness parameter  $\lambda_2$ , to check with Alice and Bob whether the state is genuinely entangled, by using the witness parameter  $\mathcal{W}_W^{\lambda_2}$  which acquires the following expectation value,

$$\text{Tr} \left[ \rho_W^{\lambda_1} \mathcal{W}_W^{\lambda_2} \right] = \frac{1}{90} \left( 35 - (23 + 42\sqrt{1 - \lambda_1^2})\lambda_2 \right). \quad (4.36)$$

Hence, Charlie<sup>2</sup> can detect genuine entanglement with Alice and Bob if  $\frac{1}{90} \left( 35 - (23 + 42\sqrt{1 - \lambda_1^2})\lambda_2 \right) < 0$ . On the other hand, from Eq.(4.34) we know that Charlie<sup>1</sup> can detect genuine entanglement with Alice and Bob when  $\lambda_1 > \frac{7}{13}$ , i.e., when  $\lambda_1 = \frac{7}{13} + \epsilon$  with  $\epsilon$  being a positive number such that  $\epsilon \leq \frac{6}{13}$ . Hence, in order to detect genuine entanglement, Charlie<sup>2</sup> must choose his sharpness parameter  $\lambda_2$  such that it satisfies  $\frac{1}{90} \left( 35 - (23 + 42\sqrt{1 - (\frac{7}{13} + \epsilon)^2})\lambda_2 \right) < 0$ . If we take  $\epsilon = 0$  (i.e.,  $\lambda_1 = \frac{7}{13}$ ), then we obtain that Charlie<sup>2</sup> can detect genuine entanglement with Alice and Bob if  $\lambda_2 > 0.60$ .

In this way if we proceed it can be observed that at most four Charlies can detect genuine entanglement through the witness operator  $\mathcal{W}_W^{\lambda_m}$  when the initial shared state is three qubit pure W-state. Allowed ranges of the sharpness parameters associated with different Charlies' measurements in order to detect genuine entanglement using the witness operator  $\mathcal{W}_W^{\lambda_m}$  are presented in Table - 4.1.

Now we are going to investigate the maximum number of Charlies that can detect genuine entanglement in the scenario mentioned in Fig. 4.1 using another type of witness operator (suitable for detecting genuine entanglement of three qubit GHZ state) which is given by [53, 144],

$$\mathcal{W}_{GHZ} = \frac{1}{2}\mathbb{I}_3 - |GHZ\rangle\langle GHZ|. \quad (4.37)$$

Now, when any Charlie<sup>m</sup> performs unsharp measurements with sharpness parameter  $\lambda_m$ , the decomposition (4.8) of the above witness operator is modified in the following way,

$$\begin{aligned} \mathcal{W}_{GHZ}^{\lambda_m} &= \frac{1}{8} \left( 3\mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I} - \mathbb{I} \otimes \sigma_z \otimes \lambda_m \sigma_z - \sigma_z \otimes \mathbb{I} \otimes \lambda_m \sigma_z - \sigma_z \otimes \sigma_z \otimes \mathbb{I} - 2\sigma_x \otimes \sigma_x \otimes \lambda_m \sigma_x \right. \\ &\quad \left. + \frac{1}{2}(\sigma_x + \sigma_y) \otimes (\sigma_x + \sigma_y) \otimes \lambda_m (\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y) \otimes (\sigma_x - \sigma_y) \otimes \lambda_m (\sigma_x - \sigma_y) \right). \end{aligned} \quad (4.38)$$

Now, since we have  $\text{Tr}[\mathcal{W}_{GHZ}\rho_{BS}] \geq 0 \forall \rho_{BS} \in \mathcal{BS}$  and  $0 < \lambda_m \leq 1$ , we can write the following:

$$\begin{aligned} &\text{Tr}[\mathcal{W}_{GHZ}^{\lambda_m}\rho_{BS}] \\ &= \lambda_m \text{Tr}[\mathcal{W}_{GHZ}\rho_{BS}] + \frac{1}{8}(1 - \lambda_m) \left( 3 - \langle \sigma_z \otimes \sigma_z \otimes \mathbb{I} \rangle \right) \\ &\geq \lambda_m \text{Tr}[\mathcal{W}_{GHZ}\rho_{BS}] + \frac{1}{4}(1 - \lambda_m) \\ &\geq 0 \quad \forall \rho_{BS} \in \mathcal{BS}. \end{aligned} \quad (4.39)$$

Hence, one may conclude that the operator  $\mathcal{W}_{GHZ}^{\lambda_m}$  after introducing unsharpness in Charlie's measurements can again be used as a valid witness operator of genuine entanglement.

In this case, consider that the three qubit GHZ state given by,  $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ , is initially shared between Alice, Bob and Charlie<sup>1</sup>. In a similar fashion described earlier,

Charlie <sup>m</sup>	Permissible ranges for $\lambda_m$
Charlie <sup>1</sup>	$1 \geq \lambda_1 > \lambda_1^{\min} = 0.33$
Charlie <sup>2</sup>	$1 \geq \lambda_2 > \lambda_2^{\min} = 0.35$ , when $\lambda_i = \lambda_i^{\min} \forall i < 2$
Charlie <sup>3</sup>	$1 \geq \lambda_3 > \lambda_3^{\min} = 0.36$ , when $\lambda_i = \lambda_i^{\min} \forall i < 3$
Charlie <sup>4</sup>	$1 \geq \lambda_4 > \lambda_4^{\min} = 0.38$ , when $\lambda_i = \lambda_i^{\min} \forall i < 4$
Charlie <sup>5</sup>	$1 \geq \lambda_5 > \lambda_5^{\min} = 0.40$ , when $\lambda_i = \lambda_i^{\min} \forall i < 5$
Charlie <sup>6</sup>	$1 \geq \lambda_6 > \lambda_6^{\min} = 0.42$ , when $\lambda_i = \lambda_i^{\min} \forall i < 6$
Charlie <sup>7</sup>	$1 \geq \lambda_7 > \lambda_7^{\min} = 0.45$ , when $\lambda_i = \lambda_i^{\min} \forall i < 7$
Charlie <sup>8</sup>	$1 \geq \lambda_8 > \lambda_8^{\min} = 0.48$ , when $\lambda_i = \lambda_i^{\min} \forall i < 8$
Charlie <sup>9</sup>	$1 \geq \lambda_9 > \lambda_9^{\min} = 0.53$ , when $\lambda_i = \lambda_i^{\min} \forall i < 9$
Charlie <sup>10</sup>	$1 \geq \lambda_{10} > \lambda_{10}^{\min} = 0.59$ , when $\lambda_i = \lambda_i^{\min} \forall i < 10$
Charlie <sup>11</sup>	$1 \geq \lambda_{11} > \lambda_{11}^{\min} = 0.67$ , when $\lambda_i = \lambda_i^{\min} \forall i < 11$
Charlie <sup>12</sup>	$1 \geq \lambda_{12} > \lambda_{12}^{\min} = 0.81$ , when $\lambda_i = \lambda_i^{\min} \forall i < 12$
Charlie <sup>13</sup>	No valid permissible range for $\lambda_{13}$ , when $\lambda_i = \lambda_i^{\min} \forall i < 13$

Table 4.2: Here we show the permissible ranges of sharpness parameters  $\lambda_m$  (where  $0 < \lambda_m \leq 1$ ) of Charlie<sup>m</sup> for detecting genuine entanglement through the witness operator  $\mathcal{W}_{GHZ}^{\lambda_m}$  with a single Alice and a single Bob at the other sides. The permissible range of each  $\lambda_m$  depends on the values  $\lambda_1, \lambda_2, \dots, \lambda_{m-1}$ . In the above table we have presented the permissible range of each  $\lambda_m$  for the minimum permissible values of  $\lambda_1, \lambda_2, \dots, \lambda_{m-1}$ . For other values of  $\lambda_1, \lambda_2, \dots, \lambda_{m-1}$ , the permissible range of each  $\lambda_m$  can also be calculated. However, the permissible ranges of  $\lambda_m$  will be smaller than that presented in the table if we take other values  $\lambda_i > \lambda_i^{\min} \forall i < m$ , and the maximum number of Charlies may get reduced. Here we find that at most twelve Charlies can detect genuine tripartite entanglement through the witness operator  $\mathcal{W}_W^{\lambda_m}$ .

we now investigate how many Charlies can detect genuine entanglement sequentially with single Alice and single Bob. Since Alice, Bob perform projective measurements and Charlie<sup>1</sup> performs unsharp measurement with sharpness parameter  $\lambda_1$ , the expectation value of the genuine entanglement witness  $\mathcal{W}_{GHZ}^{\lambda_1}$  becomes,

$$\text{Tr} \left[ |GHZ\rangle\langle GHZ| \mathcal{W}_{GHZ}^{\lambda_1} \right] = \frac{1}{4}(1 - 3\lambda_1). \quad (4.40)$$

Hence, it is clear from the above expectation value that Charlie<sup>1</sup> can detect genuine entanglement using the genuine entanglement witness  $\mathcal{W}_{GHZ}^{\lambda_1}$  with Alice and Bob when  $\lambda_1 > \frac{1}{3} \equiv 0.33$ .

After Charlie<sup>1</sup>'s unsharp measurement, the average state shared between Alice, Bob and Charlie<sup>2</sup> becomes

$$|GHZ\rangle\langle GHZ| \rightarrow \rho_{GHZ}^{\lambda_1} = \frac{1}{4} \sum_{i, \hat{z}_k^1} \left( \mathbb{I} \otimes \mathbb{I} \otimes \sqrt{E_{i|\hat{z}_k^1}^{\lambda_1}} \right) |GHZ\rangle\langle GHZ| \left( \mathbb{I} \otimes \mathbb{I} \otimes \sqrt{E_{i|\hat{z}_k^1}^{\lambda_1}} \right), \quad (4.41)$$

where  $i \in \{+1, -1\}$ ,  $\hat{z}_k^1 \in \left\{ \hat{z}, \hat{x}, \frac{\hat{x} + \hat{y}}{\sqrt{2}}, \frac{\hat{x} - \hat{y}}{\sqrt{2}} \right\}$ .

Next, Charlie<sup>2</sup> performs unsharp measurements on his part of  $\rho_{GHZ}^{\lambda_1}$  with sharpness parameter  $\lambda_2$ , to check with Alice and Bob whether the state is genuinely entangled. In this case, the expectation value of the witness operator  $\mathcal{W}_{GHZ}^{\lambda_2}$  becomes,

$$\text{Tr} \left[ \rho_{GHZ}^{\lambda_1} \mathcal{W}_{GHZ}^{\lambda_2} \right] = \frac{1}{4} \left[ 1 - \left( 1 + 2\sqrt{1 - \lambda_1^2} \right) \lambda_2 \right]. \quad (4.42)$$

Hence, Charlie<sup>2</sup> can detect genuine entanglement with Alice and Bob using the above witness if  $\frac{1}{4} \left[ 1 - \left( 1 + 2\sqrt{1 - \lambda_1^2} \right) \lambda_2 \right] < 0$ . Since, Charlie<sup>1</sup> can detect genuine entanglement with Alice and Bob when  $\lambda_1 = \frac{1}{3} + \epsilon$  with  $\epsilon$  being a positive number such that  $\epsilon \leq \frac{2}{3}$ . Hence, for detecting genuine entanglement, Charlie<sup>2</sup> must choose his sharpness parameter  $\lambda_2$  such that  $\frac{1}{4} \left[ 1 - \left( 1 + 2\sqrt{1 - \left( \frac{1}{3} + \epsilon \right)^2} \right) \lambda_2 \right] < 0$ . For example, if we take  $\epsilon = 0$  (i.e.,  $\lambda_1 = \frac{1}{3}$ ), then Charlie<sup>2</sup> can detect genuine entanglement with Alice and Bob if  $\lambda_2 > 0.35$ .

Next, we continue exploring the possibility for subsequent Charlies (Charlie<sup>3</sup>, Charlie<sup>4</sup>, ...) to detect genuine entanglement. We observe that at most twelve Charlies can detect genuine entanglement through the witness operator  $\mathcal{W}_{GHZ}^{\lambda_m}$  when the initial shared state is three qubit pure GHZ state. Allowed ranges of the sharpness parameters associated with different Charlies' measurements in order to detect genuine entanglement using the witness operator  $\mathcal{W}_{GHZ}^{\lambda_m}$  are presented in Table - 4.2.

#### 4.5 SUMMARY AND OUTLOOK

There exist several communication and computational tasks where multipartite quantum correlations serve as resources [99–101, 106, 158–167]. However, due to the difficulties present in experimentally producing multipartite quantum correlations, their implementation as pow-

erful resources in various information processing tasks are still elusive. Hence, exploring the possibilities of using single multipartite quantum correlation several times is not only interesting for foundational studies but may also be useful for information theoretic applications.

Here, we address the question as to whether multiple observers can detect genuine tripartite entanglement sequentially. We consider the scenario where three spin- $\frac{1}{2}$  particles are spatially separated and shared between, say, Alice, Bob and multiple Charlies. Alice measures on the first particle; Bob measures on the second particle and multiple Charlies measure on the third particle sequentially. In the course of our study we have used both linear as well as non-linear correlation inequalities which detect genuine entanglement in the device-independent scenario. In this context, we have shown that at most two Charlies can detect genuine entanglement of the GHZ-state. Note that the question of sharing of genuine entanglement of the W-state in the device-independent scenario remains to be investigated due to the lack of a suitable inequality. A possible direction in this context may be to evaluate the bi-separable bounds of the inequalities presented in [168], which are violated maximally by the W state.

The number of Charlies may be increased by giving up the requirement of device-independence, as we have shown using two types of appropriate genuine entanglement witness operators. Here, we find that at most four Charlies can detect genuine entanglement sequentially with the single Alice and single Bob using the shared W-state. In case of the shared GHZ-state we find that the number of Charlies can increase up to twelve, which may open up interesting possibilities of detection of genuine tripartite entanglement sharing by multiple observers.

Before concluding, it may be noted that the issue of sharing genuine nonlocality in the above scenario has been studied earlier [129]. Hence, it would be interesting to investigate this issue in the intermediate context between entanglement and Bell nonlocality, *viz.*, sharing of genuine multipartite quantum steering [169–171] by multiple observers measuring sequentially on the same particle. Finally, exploring information theoretic applications of the present study is another direction for future research.

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## DETECTING NON-MARKOVIANITY VIA UNCERTAINTY RELATION

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### 5.1 INTRODUCTION

An open quantum system, as discussed in the chapter 2, describes general quantum evolution of a quantum system interacting with its environment. As a result of this interaction with the noisy environment, dissipation, decoherence and several other irreversible phenomenon take place [23, 24, 54–58] and the system monotonically relaxes to the thermal equilibrium or more generally to a non-equilibrium steady state. These types of evolution fall under Markovian dynamics and can be described by divisible quantum maps [24, 57, 58] where information flow is a one way traffic. However, when the system-environment coupling is not sufficiently weak or the environment is non-stationary, indivisible dynamics can also occur, providing signatures of non-Markovian information back flow [25, 61–64, 172].

In recent years, it has been realized that non-Markovianity (NM) acts as a resource in various quantum mechanical tasks. For example, NM allows perfect teleportation with mixed states [26], efficient entanglement distribution [28], improvement of capacity for long quantum channels [27], and efficient work extraction from an Otto cycle [29]. For these cases, the accomplishment of the concerned tasks are done by harnessing information backflow, which can be understood as resource inter-conversion. NM can thus be inter-converted via information backflow, into other resources such as entanglement, coherent information and extractable work. It can also be exploited for efficient quantum control [173]. In view of the emerging significance of NM in quantum information processing, it has become important

to specify whether a given system-environment dynamics possesses non-Markovian traits. In other words, in order to use NM as a resource, one first needs to detect the signatures of NM. Thus, formulation of detection schemes of NM is highly relevant in a spirit similar to the construction of witness theories of other resources such as quantum entanglement and coherence. In this chapter, an experimentally feasible detection scheme has been provided for non-Markovian dynamics which is based on our paper [174].

Although, there are different approaches in defining non-Markovianity in the quantum domain [25, 61, 62, 172] including a recent attempt to construct a generalized description of NM by process tensor formalism [175], here we propose an avenue for detecting the signatures of NM, based on the definition of indivisibility of the dynamical map by interlinking it with a fundamental feature of quantum mechanics, *viz.*, uncertainty relations. This specific approach allows us to exploit the non-complete positivity spawning from the indivisibility of the channel, which leads to the breaking of uncertainty relations providing a scope to detect such operations. Our motivation thus stems from the feasibility of experimental realization of uncertainty relations between Hermitian operators, by restricting ourselves within only indivisible quantum operations.

Quantum theory restricts the accuracy of simultaneous measurements, which is well explained by the famous Heisenberg uncertainty relation [73]. A stronger lower bound of uncertainty was obtained by Robertson-Schrödinger by including an anti-commutator for more generalized pairs of observables [176]. However, for quantum states that are eigenstates of one of the observables, a non-trivial lower bound reflecting incompatibility of the observables may still be obtained by using uncertainty relations based on sums of variances [80, 81, 83]. Other forms of uncertainty relations such as entropic uncertainty relation [74, 177], and fine-grained uncertainty relation [22] have also been derived. Experimental investigations of different uncertainty relations have been performed [178–188].

Uncertainty relations have wide range of applicability. They can be used to justify the complex structure of the Hilbert space [189, 190], for detecting purity [191], entanglement [192–200], demonstration of EPR-steering [47, 67, 71, 79], security analysis of quantum key distribution [201], etc. Drawing inspiration from the literature on the usage of uncertainty relations to witness quantum correlations, we investigate the possibility of detecting non-

Markovianity through uncertainty relations. In a realistic scenario the overall dynamics is always completely-positive (CP) and hence, uncertainty relations will always hold for the overall dynamics. However, complete-positivity may break down within the intermediate time step for NM evolution. This leads to the violation of uncertainty relations within those intermediate time steps and hence, can be used to detect NM. In the present study, we propose the utility of the Robertson-Schrödinger (RS) uncertainty relation to detect NM.

This chapter is organized as follows. In the next section, we construct our theory of non-Markovianity detection via the RS uncertainty violation proposing a linear witness of NM and verify our theory with some examples. We further present a nonlinear extension of the NM witness and show that it could lead to the detection of NM dynamics beyond that possible by the linear witness. Then, we discuss a special case of unital dynamics for qubits and propose another method of detecting non-Markovianity through the dynamics of Robertson-Schrödinger uncertainty. Finally, We conclude with a summary of our results and possible implications of our approach.

## 5.2 UNCERTAINTY RELATIONS FOR NON-MARKOVIANITY DETECTION

Let's begin with a brief overview of the preliminary background of indivisible non-Markovian operations and their characterization. In chapter 2, the definition of a quantum channel and its complete-positive (CP) divisibility condition has been introduced. For the sake of completeness, here we briefly mention the condition for CP-divisibility of a quantum channel. A quantum channel is nothing but a completely positive trace preserving map,  $\Lambda_{\mathcal{N}}(t, t_0) : \rho(t_0) \rightarrow \rho(t)$ . The channel-state duality [59, 60] guarantees that the set of all quantum channels is isomorphic to that of the set of Choi-states, where the Choi-state for any quantum evolution  $\Lambda_{\mathcal{N}}(t, t_0)$  is defined as

$$\mathcal{C}_{\mathcal{N}}(t, t_0) = \mathbb{I} \otimes \Lambda_{\mathcal{N}}(t, t_0)(|\phi\rangle\langle\phi|). \quad (5.1)$$

Here  $|\phi\rangle$  is a maximally entangled state of dimension  $d \times d$  for a  $d$  dimensional system. According to Choi-Jamiolkowski isomorphism [202] for complete-positivity of the dynamics

$\Lambda_{\mathcal{N}}(t, t_0)$ , it is sufficient to show that the corresponding Choi state  $\mathcal{C}_{\mathcal{M}}(t, t_0) \geq 0, \forall(t, t_0)$ . (Throughout the study, for Choi states we use  $\mathcal{C}_{\mathcal{N}}$  and  $\mathcal{C}_{\mathcal{M}}$  interchangeably to mean the same, unless specified explicitly.)

The channel or dynamical map is called CP-divisible iff it can always be written in the form  $\Lambda_{\mathcal{N}}(t, t_0) = \Lambda_{\mathcal{N}}(t, t_1) \circ \Lambda_{\mathcal{N}}(t_1, t_0)$  where  $t > t_1 > t_0 \forall t, t_1, t_0$ . Following Rivas et al. [25], we shall use the term ‘Markovian Choi-state’ if the corresponding dynamics is CP-divisible. Rest of the Choi states, corresponding to indivisible channels are considered as non-Markovian Choi states.

A dynamics of Lindblad form is,  $\dot{\rho}(t) = \mathcal{L}_t(\rho(t))$  with  $\mathcal{L}_t(\cdot) = \sum_i \Gamma_i(t)(L_i(\cdot)L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i, (\cdot)\})$ . Here  $\Gamma_i(t)$ ’s are Lindblad coefficients and  $L_i$ ’s are Lindblad operators. A necessary and sufficient condition for CP-divisibility of the dynamics is that  $\Gamma_i(t) \geq 0 \forall(i, t)$ , [56].

However, in many practical situations CP-divisibility may break down, leading to the observation of non-Markovian backflow of information. This approach of non-Markovianity via CP-indivisibility has been used extensively in the recent literature [24, 61–64, 172]. It is evident from these studies that CP-indivisibility of arbitrary dynamical evolutions leads to the violation of positive semi-definiteness of the corresponding Choi states [59, 60]. This leads us to the idea of detecting non-Markovianity by the violation of uncertainty relations.

### 5.2.1 Violation of Uncertainty relations

Robertson [176] showed that uncertainty relations can be found using the property of positive semi-definiteness of a quantum state. More recently it has been proven [194] that positive semi-definiteness of a state is necessary for uncertainty relations to hold. For example, in a two-dimensional Hilbert space a state is described by a  $2 \times 2$  matrix  $\rho_1 = \begin{pmatrix} a & c \\ c^* & b \end{pmatrix}$  such that  $Tr[\rho_1] = 1$  and  $Det[\rho_1] = ab - |c|^2 \geq 0$ .

The Robertson-Schrödinger (RS) uncertainty relation for two observables  $A$  and  $B$  (in arbitrary dimensions) is given by,

$$\Delta^2 A \Delta^2 B - \frac{1}{4} |\langle [A, B] \rangle|^2 - \frac{1}{4} |(\langle \{A, B\} \rangle - 2\langle A \rangle \langle B \rangle)|^2 \geq 0. \quad (5.2)$$

Here the variance of  $A$  (similarly for  $B$ ) is defined as  $\Delta^2 A \equiv \langle A^2 \rangle - \langle A \rangle^2$ , and all expectation values are calculated over some quantum state  $\rho$ . The RS relation holds if  $\rho$  is non-negative.

For a CP divisible evolution  $\mathcal{M}$ , the corresponding Choi state  $\mathcal{C}_{\mathcal{M}}(t + \epsilon, t) \geq 0, \forall(t, \epsilon)$ , the uncertainty relations will always hold. However, for some arbitrary CP-indivisible evolution  $\mathcal{N}$ ,  $\mathcal{C}_{\mathcal{N}}(t + \epsilon, t)$  can acquire negative eigenvalues within some time interval  $\epsilon$  and may lead to the violation of uncertainty relations. We use this fact to state and prove the following theorem.

**Theorem 1:** *The statement of this theorem is twofold.*

- 1) *Violation of the Robertson-Schrödinger uncertainty relation for the Choi state of a dynamical evolution is sufficient to detect CP-indivisibility.*
- 2) *There always exists a set of Hermitian operators, for which RS uncertainty relation will be violated in case of CP-indivisible operations.*

**Proof.** Let us consider the derivation [203] of RS-uncertainty relation given by (5.2). In order to do that, we evaluate  $\text{Tr}[\mathcal{C}_{\mathcal{N}}TT^\dagger]$  for any operator  $T = A_0 + (\gamma + i\epsilon)B_0$ , where  $A_0 = A - \langle A \rangle$  and  $B_0 = B - \langle B \rangle$ .  $\mathcal{C}_{\mathcal{N}}$  is an arbitrary Choi state over which the averages are calculated. It is easy to show that  $TT^\dagger$  is always positive semi-definite for any operator  $T$ . After a few steps of simplification, we get

$$\text{Tr}[\mathcal{C}_{\mathcal{N}}TT^\dagger] = \text{Tr}[\mathcal{C}_{\mathcal{N}}A_0^2] + (\gamma^2 + \epsilon^2)\text{Tr}[\mathcal{C}_{\mathcal{N}}B_0^2] + \gamma\text{Tr}[\mathcal{C}_{\mathcal{N}}\{A_0, B_0\}] - i\epsilon\text{Tr}[\mathcal{C}_{\mathcal{N}}[A_0, B_0]].$$

One can find that the value of  $\text{Tr}[\mathcal{C}_{\mathcal{N}}TT^\dagger]$  is minimum for  $\gamma = -\frac{\text{Tr}[\mathcal{C}_{\mathcal{N}}\{A_0, B_0\}]}{2\text{Tr}[\mathcal{C}_{\mathcal{N}}B_0^2]}$  and  $\epsilon = \frac{i\text{Tr}[\mathcal{C}_{\mathcal{N}}[A_0, B_0]]}{2\text{Tr}[\mathcal{C}_{\mathcal{N}}B_0^2]}$ . Substituting these, one has

$$\begin{aligned} \min_{\gamma, \epsilon} \text{Tr}[\mathcal{C}_{\mathcal{N}}TT^\dagger] &= \frac{1}{\text{Tr}[\mathcal{C}_{\mathcal{N}}B^2] - (\text{Tr}[\mathcal{C}_{\mathcal{N}}B])^2} [(\text{Tr}[\mathcal{C}_{\mathcal{N}}A^2] - (\text{Tr}[\mathcal{C}_{\mathcal{N}}A])^2)(\text{Tr}[\mathcal{C}_{\mathcal{N}}B^2] - (\text{Tr}[\mathcal{C}_{\mathcal{N}}B])^2) \\ &\quad + \frac{1}{4}(\text{Tr}[\mathcal{C}_{\mathcal{N}}[A, B]])^2 - \frac{1}{4}((\text{Tr}[\mathcal{C}_{\mathcal{N}}\{A, B\}] - 2\text{Tr}[\mathcal{C}_{\mathcal{N}}A]\text{Tr}[\mathcal{C}_{\mathcal{N}}B]))^2] \end{aligned}$$

which represents the left hand side of (5.2).

Now since  $TT^\dagger$  is always positive; the identity,

$$\text{Tr}[\mathcal{C}_{\mathcal{N}}TT^\dagger] \geq 0 \tag{5.3}$$

always holds, only if  $\mathcal{C}_{\mathcal{N}}$  is positive semi-definite. Using the above identity one can have (5.2). Since  $TT^\dagger$  is always positive, the identity can break only if  $\mathcal{C}_{\mathcal{N}}$  is negative. In other words, the violation of the uncertainty relation is sufficient to know that  $\mathcal{C}_{\mathcal{N}}$  is not positive semi-definite, and hence, the dynamics is indivisible.

However, in order to show that violation of the uncertainty relation is not necessary to detect CP-indivisibility, consider that  $\mathcal{C}_{\mathcal{N}}$  has a spectral decomposition of the form,

$$\mathcal{C}_{\mathcal{N}} = \sum_i \lambda_i^+ |\lambda_i^+\rangle\langle\lambda_i^+| + \sum_j \lambda_j^- |\lambda_j^-\rangle\langle\lambda_j^-|, \quad (5.4)$$

where  $\lambda_i^+$  and  $\lambda_j^-$  are the positive and negative eigenvalues respectively. In this case there exists an operator  $TT^\dagger$  such that  $TT^\dagger$  is orthogonal to the eigen-basis corresponding to negative eigenvalues of  $\mathcal{C}_{\mathcal{N}}$  (for example  $TT^\dagger = |\lambda_i^+\rangle\langle\lambda_i^+|$ ). In this situation even if  $\mathcal{C}_{\mathcal{N}}$  is negative, the uncertainty relation may still hold.

Now, to prove the second part of the theorem that there always exists a pair of Hermitian operators which will violate the RS uncertainty relation for CP-indivisible Choi states, let us consider the Choi-state in its spectral decomposition form (5.4). The trace preservation condition yields  $\sum_i \lambda_i^+ + \sum_j \lambda_j^- = 1$ . This shows us that for any arbitrary choice for the non-Markovian Choi state, there exists at least one positive eigenvalue, to preserve this trace relation. Let us consider one of the Hermitian operators to be  $H_1 = |\lambda_j^-\rangle\langle\lambda_j^-|$ , which will give us the negative eigenvalue  $\lambda_j^-$ , if acted upon the Choi state  $\mathcal{C}_{\mathcal{N}}$ . For this, the variance will give us  $\Delta^2 H_1 = \lambda_j^- - (\lambda_j^-)^2 < 0$ . Let us now consider another Hermitian operator  $H_2 = \sum_{k \neq l} |\lambda_k\rangle\langle\lambda_l|$ , where  $|\lambda_k\rangle$  and  $|\lambda_l\rangle$  can take any values from the spectral decomposition of  $\mathcal{C}_{\mathcal{N}}$  with corresponding eigen values  $\lambda_k$  and  $\lambda_l$ . If there are  $n$  number of terms in the spectral decomposition of  $\mathcal{C}_{\mathcal{N}}$  then  $\langle H_2^2 \rangle = (n-1) \sum_k \lambda_k = n-1$ ,  $\langle H_2 \rangle = 0$  and hence  $\Delta^2 H_2 > 0$ . Therefore, the term  $\Delta^2 H_1 \Delta^2 H_2 < 0$ , making the left hand side of RS relation (5.2) to be negative, for the pair of operators  $H_1$  and  $H_2$ . For any given Choi state, which is CP-indivisible, i.e., containing at least one negative eigenvalue, we can always find such a pair of  $H_1$  and  $H_2$ . This proves the theorem.

Recently, it has been shown that one can have convex compact structure of the set of Markovian-Choi states if we restrict ourselves only to those dynamical maps which have

Lindblad type generators and in short time interval  $(t, t + \epsilon)$  approximation ( $\epsilon$  is sufficiently small) [204]. In the following proposition, using convex structure of the set of Markovian-Choi states, we discuss another uncertainty like relation, whose violation will sufficiently detect non-Markovian dynamics.

**Proposition 1:** For some observable  $A_i$  and some state  $\mathcal{C}_{\mathcal{M}}$  written in a convex combination [i.e.  $\mathcal{C}_{\mathcal{M}} = \sum_k p_k \mathcal{C}_{\mathcal{M}}^k$  where  $p_k \geq 0, \sum_k p_k = 1$ ] of some Markovian Choi-states,  $\mathcal{C}_{\mathcal{M}}^k$  within the set of all Markovian Choi-states having Lindblad type generators ( $\mathcal{F}_{\mathcal{M}}^\epsilon$ ),

$$\sum_i \Delta^2(A_i)_{\mathcal{C}_{\mathcal{M}}} \geq \sum_k p_k \sum_i \Delta^2(A_i)_{\mathcal{C}_{\mathcal{M}}^k}, \quad (5.5)$$

holds, where  $\Delta^2(A_i)_X$  is the variance of  $A_i$  over the state  $X$  as defined earlier. We call a state "violating proposition 1" if and only if there are no states  $\mathcal{C}_{\mathcal{M}}^k \in \mathcal{F}_{\mathcal{M}}^\epsilon$  and no  $p_k$  such that above equation is fulfilled.

**Proof.** This fact is known from a different perspective of entanglement theory [192, 193].

For each  $A_i$ ,

$$\begin{aligned} \Delta^2(A_i)_{\mathcal{C}_{\mathcal{M}}} &= \sum_k p_k \langle (A_i - \langle A_i \rangle_{\mathcal{C}_{\mathcal{M}}})^2 \rangle_{\mathcal{C}_{\mathcal{M}}} \\ &= \sum_k p_k (\langle A_i^2 \rangle_{\mathcal{C}_{\mathcal{M}}^k} - \langle A_i \rangle_{\mathcal{C}_{\mathcal{M}}^k}^2 + \langle A_i \rangle_{\mathcal{C}_{\mathcal{M}}^k}^2 - 2\langle A_i \rangle_{\mathcal{C}_{\mathcal{M}}^k} \langle A_i \rangle_{\mathcal{C}_{\mathcal{M}}} + \langle A_i \rangle_{\mathcal{C}_{\mathcal{M}}}^2) \\ &= \sum_k p_k (\Delta^2(A_i)_{\mathcal{C}_{\mathcal{M}}^k} + (\langle A_i \rangle_{\mathcal{C}_{\mathcal{M}}^k} - \langle A_i \rangle_{\mathcal{C}_{\mathcal{M}}})^2) \\ &\geq \sum_k p_k \Delta^2(A_i)_{\mathcal{C}_{\mathcal{M}}^k}. \end{aligned}$$

Now summing over  $i$ , we get

$$\sum_i \Delta^2(A_i)_{\mathcal{C}_{\mathcal{M}}} \geq \sum_k p_k \sum_i \Delta^2(A_i)_{\mathcal{C}_{\mathcal{M}}^k},$$

Since the proposition is based on convex structure of the set of Markovian Choi-states  $\mathcal{C}_{\mathcal{M}}$ , violation of this proposition implies the state  $\mathcal{C}_{\mathcal{M}}$  lies outside the convex set and hence must be non-Markovian. In fact this proposition is a sufficient criterion, if we choose  $A_i$  appropriately.

One may be interested to obtain the lower bound of the above derived uncertainty inequality. From the above equation it is clear that the equality holds for pure Markovian

Choi-states, i.e., if the state  $\mathcal{C}_{\mathcal{M}}$  does not have any decomposition other than itself. Next, one has to optimize over all such pure Markovian Choi-states.

In the following proposition, we consider another uncertainty relation, which can be advantageous than the RS uncertainty relation to detect non-Markovianity.

**Proposition 2:** For two Hermitian operators  $A$  and  $B$ ,

$$\Delta^2 A + \Delta^2 B \geq |\langle [A, B] \rangle|, \quad (5.6)$$

where  $\langle (\cdot) \rangle = \text{Tr}((\cdot)\mathcal{C}_{\mathcal{M}})$ , with  $\mathcal{C}_{\mathcal{M}} \in \mathcal{F}_{\mathcal{M}}^e$ .

**Proof.** The proof is straightforward. We know  $(\Delta A - \Delta B)^2 \geq 0$  or  $\Delta^2 A + \Delta^2 B \geq 2\Delta A \Delta B$ . Now, the Heisenberg uncertainty relation for  $A$  and  $B$  is given by  $\Delta A \Delta B \geq \frac{1}{2} |\langle [A, B] \rangle|$ .

Here we observe that choosing a suitable observable is important, so that the expectation value of the observable corresponding to some non-Markovian Choi-state  $\mathcal{C}_{\mathcal{N}}$  becomes negative. Therefore, violation of “**Proposition 2**” implies detection of non-Markovianity. The reason to propose this sum uncertainty relation is due to the fact that it can detect non-Markovianity through its violation, in cases where the product uncertainty relation like the RS uncertainty do not show violation. Let us elucidate more on this issue of non-Markovianity detection in the context of the following example.

Consider a non-Markovian dynamics having Choi state  $\mathcal{C}_{\mathcal{N}}$ , which has at least two negative eigenvalues  $\lambda_1^-$  and  $\lambda_2^-$ , corresponding to the eigen-vectors  $|\lambda_1^- \rangle, |\lambda_2^- \rangle$  respectively. We construct two projectors  $W_1 = |\lambda_1^- \rangle \langle \lambda_1^- |$  and  $W_2 = |\lambda_2^- \rangle \langle \lambda_2^- |$ . The RS uncertainty (5.2) for these two observables, gives us  $\lambda_1^- \lambda_2^- (1 - (\lambda_1^- + \lambda_2^-)) \geq 0$ . Clearly, equation (5.2) is not violated since  $\lambda_1^-$  and  $\lambda_2^-$  are both negative. However, if one considers the sum uncertainty (5.6), the LHS becomes negative, but the RHS being commutator of two orthogonal projectors vanishes. We can see that the violation of equation (5.6) occurs for this case. This example shows that the sum uncertainty relation (5.6) can be advantageous compared to product uncertainty relations for the detection of non-Markovianity.

### 5.2.2 Examples

In this sub-section, we present various practical examples from quantum channels, to validate the theory we constructed in the above presentation.

*Dephasing channel:* The Lindblad equation for a pure dephasing channel is given by,

$$\frac{d\rho}{dt} = \gamma(t)(\sigma_z \rho \sigma_z - \rho),$$

where  $\sigma_z$  is the z component of Pauli matrix and  $\gamma(t)$  is defined as,

$$\gamma(t) = \frac{2\lambda\gamma_0 \sinh[tg/2]}{g \cosh[tg/2] + \lambda \sinh[tg/2]}.$$

with  $g = \sqrt{\lambda^2 - 2\gamma_0\lambda}$ . It has been shown that the dynamics will be non-Markovian when  $\gamma(t)$  has a negative region and that happen only when  $\gamma_0 > \lambda/2$  [205]. Here we calculate the LHS of RS-uncertainty relation (Eq. (5.2))  $R(t)$  for the Choi-state of this dephasing channel under the small time approximation ( $\epsilon\gamma(t) \ll 1$ ), and observe that uncertainty relations get violated only when the Lindblad coefficient  $\gamma(t)$  is negative. This confirms that violation of uncertainty relation is sufficient to show that the dynamics is non-Markovian. Fig. 5.1 depicts the uncertainty profile as a function of time for the dephasing channel for the observables  $S_x = \sigma_x \otimes \sigma_y$ ,  $S_y = \sigma_x \otimes \sigma_x$ . Note that, in figure we consider dimensionless quantity  $\bar{t} = t/\kappa$  and  $\bar{\Gamma}(t) = \kappa\gamma(t)$  for time and Lindblad coefficient respectively, where  $\kappa$  is a constant of the dimension of time.

*Spin bath model:* Here we use the generalized non-Markovian evolution for a spin-bath model. This type of system has drawn significant interest in recent studies [206–209]. The model considered in the present study consists of a central spin interacting uniformly with a collection of mutually non-interacting and completely unpolarized spins, considered as the environment. An exact canonical master equation of the Lindblad form for such a model has been derived earlier [210]. From the master equation of the Lindblad form it was shown that

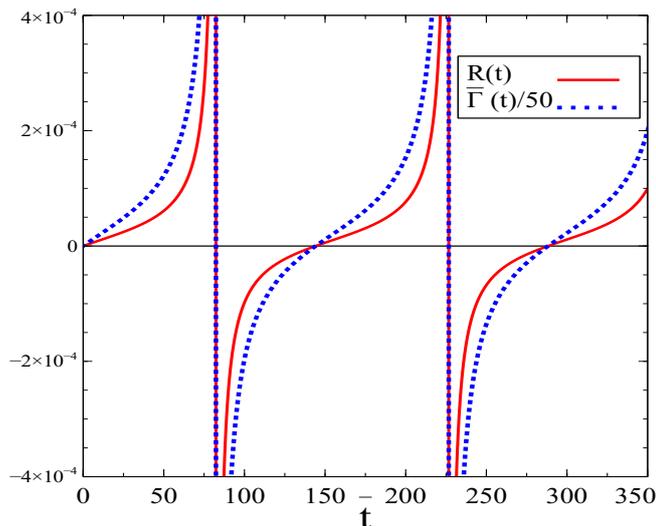


Figure 5.1: Plot for the RS-uncertainty quantity  $R(t)$  for a Choi-state and the Lindblad-coefficient ( $\bar{\Gamma}(t)$ ) with dimensionless time  $t$  for the Choi-state of the dephasing channel. Clearly, the RS-uncertainty relation is violated (i.e,  $R(t) < 0$ ) only where  $\bar{\Gamma}(t)$  is negative.

the reduced dynamics of the central spin exhibits non-Markovian features throughout the evolution. The dynamical equation for the mentioned reduced evolution is given by

$$\begin{aligned} \dot{\rho}(t) = & \frac{i}{\hbar} U(t) [\rho(t), \sigma_z] + \Gamma_{deph}(t) [\sigma_z \rho(t) \sigma_z - \rho(t)] + \Gamma_{dis}(t) \left[ \sigma_- \rho(t) \sigma_+ - \frac{1}{2} \{ \sigma_+ \sigma_-, \rho(t) \} \right] \\ & + \Gamma_{abs}(t) \left[ \sigma_+ \rho(t) \sigma_- - \frac{1}{2} \{ \sigma_- \sigma_+, \rho(t) \} \right], \end{aligned} \quad (5.7)$$

where  $\sigma_{\pm} = \frac{\sigma_x \pm i\sigma_y}{2}$  and  $\Gamma_{dis}(t), \Gamma_{abs}(t), \Gamma_{deph}(t)$  are the rates of dissipation, absorption, dephasing processes respectively, and  $U(t)$  corresponds to the unitary evolution. The uncertainty profile for this particular evolution is depicted in Fig. 5.2, showing the detection of non-Markovianity through violation of uncertainty. Note that, similar to the case of dephasing channel, all the quantities in the plot are made dimensionless, by scaling with the parameter  $\kappa$ .

### 5.2.3 Variance as a Non-linear witness of non-Markovianity

Recently, it has been shown that for dynamical maps having Lindblad type generators, one can have a resource theory of non-Markovianity where divisible operations are the free op-

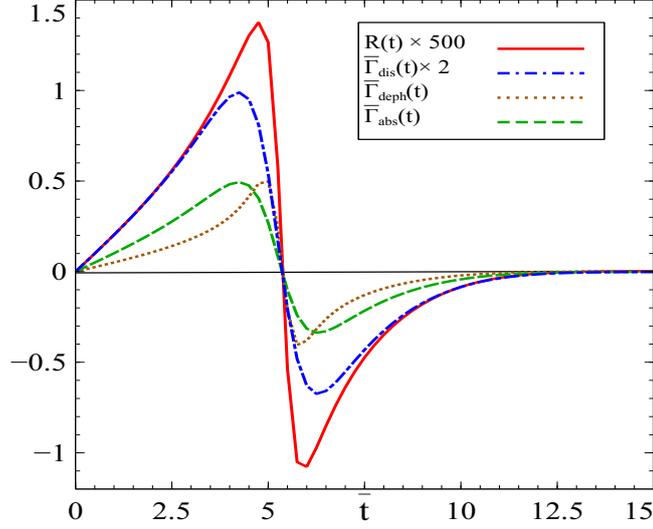


Figure 5.2: Plot for RS-uncertainty relation ( $R(t)$ ) for a Choi-state and various dimensionless Lindblad-coefficients ( $\bar{\Gamma}_{dis}(t)$ ,  $\bar{\Gamma}_{abs}(t)$ ,  $\bar{\Gamma}_{deph}(t)$ ) with time  $\bar{t}$  for the Choi-state of spin-bath model. Clearly RS-uncertainty relation is violated (i.e,  $R(t) < 0$ ) only at those regions where at least one of the Lindblad-coefficients is negative.

erations and the Choi-states corresponding to the free operations are the free states [204]. It was shown that for a sufficiently small  $\epsilon$ , the set of all Markovian Choi-states having Lindblad type generators will take a convex and compact form ( $\mathcal{F}_{\mathcal{M}}^{\epsilon}$ ). In a following work [211], a theory of linear witnesses was further established for detecting non-Markovianity. It was shown that the set of Choi states for divisible operations does not form a polytope. Consequently, we can surmise that linear witnesses will not be sufficient for optimal detection of non-Markovianity, and hence, non-linear corrections of linear witnesses are in order. The possibility of detecting non-Markovianity via violation of uncertainty relations gives us one such opportunity. As we have mentioned earlier, one can construct NM witness [211] by utilizing the fact that the Markovian Choi states having Lindblad generators form a convex and compact set ( $\mathcal{F}_{\mathcal{M}}^{\epsilon}$ ) under the small time interval approximation [204]. A Hermitian operator  $W$  is said to be a non-Markovian witness if the following criteria hold:

1.  $Tr(W\mathcal{C}_{\mathcal{M}}) \geq 0 \forall \mathcal{C}_{\mathcal{M}} \in \mathcal{F}_{\mathcal{M}}^{\epsilon}$ ,
2. There exists at least one NM Choi-state  $\mathcal{C}_{\mathcal{N}}$  such that  $Tr(W\mathcal{C}_{\mathcal{N}}) < 0$ .

Let  $\mathcal{C}_{\mathcal{M}}$  be a Choi-state. Since  $\mathcal{C}_{\mathcal{M}}$  is Hermitian, one can have it's spectral decomposition of the form

$$\mathcal{C}_{\mathcal{M}} = \sum_i \lambda_i P_i,$$

where  $P_i$ s are orthogonal projectors corresponding to the eigenvalues  $\lambda_i$ . If the state is Markovian (CP-divisible) then  $\mathcal{C}_M$  is a valid state as it has all non-negative eigenvalues. Hence,  $\text{Tr}(\mathcal{C}_M P_i) \geq 0, \forall i$ . However, if the operation is non-Markovian, then  $\text{Tr}(\mathcal{C}_N P_i) < 0$  for at least one  $i$ . So, orthogonal projectors serve as witness.

**Proposition 3:** *Corresponding to a projective NM witness  $W_i$ , it's variance  $\Delta^2 W_i$  is also a witness.*

**Proof.** The variance of  $W_i$  over some state  $\mathcal{C}_N$  as defined earlier is  $\Delta^2 W_i = \text{Tr}[\mathcal{C}_N \cdot W_i^2] - (\text{Tr}[\mathcal{C}_N \cdot W_i])^2$ . Since  $W_i$  is a projector corresponding to the eigenvalue say  $\lambda_i$ ,  $W_i = |\lambda_i\rangle\langle\lambda_i|$ , so  $W_i^2 = W_i = |\lambda_i\rangle\langle\lambda_i|$ . If  $\mathcal{C}_N$  is Markovian, then it has all non negative eigenvalues. We know that variance for such a state cannot be negative, i.e.,

$$\Delta^2(W_i) \geq 0, \forall \mathcal{C}_N \in \mathcal{F}_M^e.$$

But if the operation is non-Markovian,  $\text{Tr}[\mathcal{C}_N W_i] < 0$  for at least one  $i$ , say for  $i = 1$ , and then the variance for the witness (projector corresponding to the negative eigenvalue) becomes

$$\Delta^2(W_1) = \lambda_1 - (\lambda_1)^2,$$

which is always negative. Therefore,  $\Delta^2(W_i)$  satisfies the conditions 1 and 2, and hence, serves as a non-linear witness.

In fact it is clear that  $\Delta^2(W_i)$  is an improvement over the actual projective witness  $W_i$ , since the variance contains an additional negative term. There is another corollary which can be drawn as a consequence of **Proposition 3**.

*Corollary:* Variance of some suitable Hermitian operator  $\mathcal{H}$  over Choi states can be considered as a non-linear witness of non-Markovianity.

**Proof.** The variance of a Hermitian operator  $\mathcal{H}$  can be expressed as

$$\Delta^2(\mathcal{H}) = \text{Tr}[\mathcal{H}^\dagger \mathcal{H} \mathcal{C}_N] - (\text{Tr}[\mathcal{H} \mathcal{C}_N])^2$$

Now, similar to the proof of **Proposition 3**, it can be shown that

1.  $\Delta^2(\mathcal{H}) \geq 0$  for all Markovian Choi states  $\mathcal{C}_{\mathcal{M}}$ ,
2. There exists at least one non-Markovian Choi state  $\mathcal{C}_{\mathcal{N}}$ ,  
for which  $\Delta^2(\mathcal{H}) < 0$ .

This proves the corollary.

### 5.3 UNCERTAINTY BASED NON-MARKOVIANITY QUANTIFIER FOR UNITAL DYNAMICS OF QUBITS

In this section we consider the case of qubit channels and unital dynamics which have been studied recently in several experimental works [178–188]. Here the RS-uncertainty function calculated over a physical state  $\rho(t) = \Lambda(t)\rho(0)$  (note that this is the time evolved state, not the Choi-state). We show that if we restrict ourselves to unital dynamics, then the RS-uncertainty quantity is a monotonically increasing function and can only decrease in the regions where CP divisibility breaks down.

**Proposition 4:** *Non-Markovianity is necessary to decrease RS-uncertainty of time-evolving physical states undergoing unital dynamics.*

**Proof.** It has been shown that the RS-uncertainty relation can be used to detect purity and mixedness of a state[191]. The RS-uncertainty relation for a physical state  $\rho$  may be written as

$$R(A, B, \rho) \geq 0, \tag{5.8}$$

where  $R(A, B, \rho)$  is defined as,

$$R(A, B, \rho) \equiv \Delta^2 A \Delta^2 B - \left| \frac{\langle [A, B] \rangle}{2} \right|^2 - \left| \left( \frac{\langle \{A, B\} \rangle}{2} - \langle A \rangle \langle B \rangle \right) \right|^2. \tag{5.9}$$

On the other hand, linear entropy is defined as

$$S_l(\rho) = \frac{d}{d-1} [1 - \text{Tr}[\rho^2]], \quad (5.10)$$

where, a qubit  $\rho$  may in general be expressed as  $\rho = \frac{\mathbb{I} + \hat{n} \cdot \hat{\sigma}}{2}$ .

By choosing  $A = \hat{r} \cdot \hat{\sigma}$  and  $B = \hat{t} \cdot \hat{\sigma}$ , it has been shown [191] that

$$R(A, B, \rho) = \left[ 1 - \left( \sum_i r_i t_i \right)^2 \right] S_l(\rho). \quad (5.11)$$

It follows from Eq[5.10] that,

$$\frac{d}{dt} S_l(\rho) = -2 \frac{d}{d-1} \text{Tr}[\rho \dot{\rho}]. \quad (5.12)$$

Further, considering unital dynamics of Lindblad operators [204, 210, 212], it can be shown that,

$$\frac{d}{dt} S_l(\rho) = \frac{d}{d-1} \sum_i \Gamma_i(t) Q_i(t), \quad (5.13)$$

where  $Q_i(t) = \| [V_i, \rho(t)] \|_{HS}^2$  with  $V_i$  the Lindblad operator for an arbitrary unital channel and  $\| X \|_{HS} = \sqrt{\text{Tr}[X^\dagger X]}$  is the Hilbert-Schmidt norm.  $Q_i(t)$  is a known measure of quantumness [213, 214]. Combining the above results, here we get

$$\frac{dR}{dt} = \frac{d}{d-1} \left[ 1 - \left( \sum_i r_i t_i \right)^2 \right] \sum_i \Gamma_i(t) Q_i(t), \quad (5.14)$$

where,  $\sum_i r_i t_i < 1$ . Since  $Q_i(t)$  is always positive, uncertainty will always increase unless  $\Gamma_i(t)$  becomes negative. Hence, non-Markovianity is necessary to decrease uncertainty for any unital dynamics of a qubit.

In order to avoid any dependency of the above expression over observables  $A$  and  $B$ , one may choose a priori orthogonal operators i.e,  $\hat{r} \cdot \hat{t} = 0$ . We plot the evolution of RS-uncertainty quantity with time in Fig. 5.3, for a initial qubit  $\rho(0) = \frac{1}{2} |+\rangle \langle +|$  (where  $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ ), evolving under unital dynamics (when  $\Gamma_{abs} = \Gamma_{dis}$ ) for the spin-bath model given by equation

(5.7) with observables  $A = \sigma_x$  and  $B = \sigma_y$ . It is clear from the plot that the RS-uncertainty decreases only when the CP-divisibility breaks down.

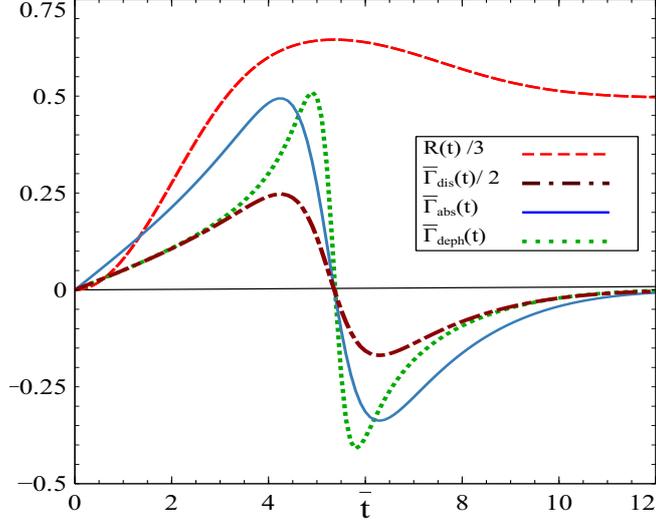


Figure 5.3: Plot for RS-uncertainty relation for a time-evolving physical state and various Lindblad-coefficients ( $\bar{\Gamma}_{dis}(t)$ ,  $\bar{\Gamma}_{abs}(t)$ ,  $\bar{\Gamma}_{deph}(t)$ ) with time  $\bar{t}$  for a qubit undergoing unital dynamics given by the spin bath model. It is seen that negativity of Lindblad-coefficient (and hence non-Markovianity) is necessary in order to decrease uncertainty for this unital dynamics. All quantities are dimensionless.

The above findings show that the decrease of  $R$  for time-evolving physical states signifies information backflow. Based on this, we propose a non-Markovian quantifier for unital qubit dynamics, which can be defined as,

$$\mathcal{N} = - \int_{\frac{dR}{dt} < 0} \frac{dR}{dt} dt. \quad (5.15)$$

This measure amounts to a certain quantification of total amount of non-Markovianity. As a further remark, choosing  $\hat{r}$  and  $\hat{t}$  as orthogonal vectors, our proposed measure of non-Markovianity turns out to be identical to the measure based on purity [210].

#### 5.4 SUMMARY AND OUTLOOK

To summarize, in this study, we develop a framework for detection of non-Markovianity using the mechanism of quantum uncertainty relations. In particular, we formulate two different ways to detect signatures of non-Markovianity in open system dynamics. First, we es-

establish that violation of uncertainty relations for Choi states corresponding to the evolution dynamics is sufficient to detect non-Markovianity. Our formalism is exemplified in the context of dephasing and spin-bath models for which non-Markovian evolution can be certified by the violation of the Robertson-Schrödinger uncertainty relation. Based on our approach, we propose certain linear and non-linear witnesses of non-Markovianity. We further present an example to clarify the advantage of the sum uncertainty relation for non-Markovianity detection. Secondly, adopting another approach of using uncertainty relations, we show that for unital dynamics non-Markovianity is necessary to decrease the RS-uncertainty for the time-evolving physical states. This enables us to propose a new non-Markovianity quantifier for unital dynamics of qubits.

Before concluding, it may be worthwhile to discuss the feasibility of experimental verification of the approaches discussed here. Variances of observables have been measured using optical set-ups in several experimental schemes [183, 187, 188]. On the other hand, there has been considerable recent activity towards experimental realization of non-Markovianity under dephasing optical channels [215–219]. In light of these results, it may be feasible to measure the RS uncertainty corresponding to the physical states undergoing unital dynamics in order to conclude whether the dynamics is non-Markovian. It is also important to mention that in order to compute the uncertainty quantity, we do not need to do a full process tomography as required for several other methods of detecting non-Markovianity [25, 61, 62, 172]. Therefore, in case of determining the nature of the specific quantum evolution experimentally, our method could be more efficient and cost effective. It is thus evident that with further development, our proposed framework should provide an avenue for experimental investigation of non-Markovian evolution in a quantitative manner.

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## APPLICATION OF INDEFINITE CAUSAL ORDER IN QUANTUM COMMUNICATION

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### 6.1 INTRODUCTION

Communication theory deals with reliable transfer of information experiencing inevitable noisy evolutions almost in all practical scenarios. Claude Shannon, in his seminal work [3], formulated a generic mathematical modeling of such noisy processes laying down the foundations of present communication era. In Shannon theory, both the information and its transfer in space-time are treated as classical. Its quantum generalization, the quantum Shannon theory [40, 220], exploits rules of quantum theory to devise innovative communication protocols. Quantum communication theory deals with the transmission of quantum messages from a sender to a receiver. When the identity of the receiver is known, this task can be addressed with a reliable quantum communication channel between the sender and the receiver. But what if the identity of the receiver is unknown as well as the channels are noisy? The present chapter which is based on one of our papers [221], addresses this question by introducing a new quantum communication task called *random-receiver quantum communication*. Surprisingly, we also show that the task can be perfectly accomplished with the assistance of indefinite causal order (introduced in chapter 2) even though there are entanglement breaking channels from sender to each receiver. Such an indefinite causal order can be obtained through a novel primitive called *quantum SWITCH* that entangles relative order of two channels with a control system prepared in quantum superposition [31]. Utilization of *quantum*

*SWITCH* have been studied in a variety of problems [30, 36–39, 65, 222–229] and its physical implementations have also been obtained in different photonic setup [34, 230–232].

## 6.2 RANDOM-RECEIVER QUANTUM COMMUNICATION

Let us now discuss the task of *random-receiver quantum communication*. A sender  $A$  is connected to  $n$  spatially separated parties  $(B_i)_{i=1}^n$  through  $n$  communication channels  $(C_i)_{i=1}^n$ , as in Fig. 6.1. The sender wants to transmit a quantum message to the  $x$ -th party, for some  $x \in \{1, \dots, n\}$ . However, the identity of such party (*i.e.* the value of  $x$ ) is unknown to the sender. This scenario could arise, for example, in delegated quantum computation, where a client sends inputs to a server, asking it to perform a desired quantum computation on them. Here the receiver  $B_x$  could be one of  $n$  servers, and the sender may not know in advance which server is available to perform the desired computation. In this situation, the sender has to delocalize the message, and send it to all the  $n$  servers, in such a way that the available one can retrieve the message and operate on it. In the following, we will assume that the  $n$  parties know the value of  $x$  (for example, because they have communicated classically among each other) and cooperate in order to let the message reach party  $B_x$ . To coordinate their actions, the parties are allowed to exchange classical messages. A communication protocol is said to be successful if it works for all values of  $x \in \{1, \dots, n\}$ .

Random-receiver quantum communication is related to the task of quantum summoning [233, 234], where a quantum message has to be revealed at a given set of spacetime points. The crucial difference, however, is that summoning includes limits on the exchange of signals among the  $n$  parties induced by the structure of the underlying spacetime. In random-receiver quantum communication, classical communication among the  $n$  parties is permitted, while quantum communication is forbidden.

To introduce the task of random-receiver quantum communication, let us first consider the simple scenario where the quantum message is a generic state of a qubit, and all the channels from the sender to the receivers are noiseless. To transmit the quantum state  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ , the sender can encode it in the generalized Greenberger-Horne-Zeilinger (GHZ) state  $|\psi_n\rangle := \alpha|0\rangle^{\otimes n} + \beta|1\rangle^{\otimes n}$  and send such state to the  $n$  receivers. To let party

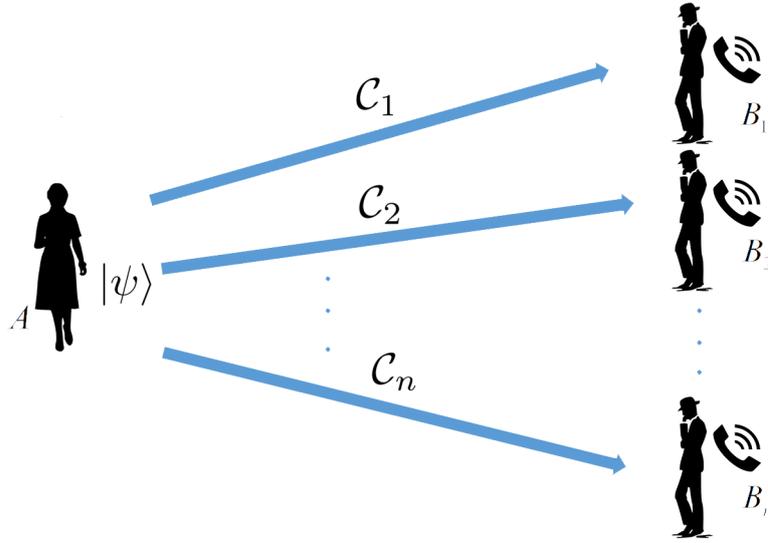


Figure 6.1: **Random-receiver quantum communication.** The sender  $A$  wants to transmit a quantum message to one of  $n$  receivers  $(B_i)_{i=1}^n$ . The identity of the designated receiver  $B_x$  is unknown at the moment of transmission. After the designated receiver  $B_x$  is revealed, the other receivers cooperate with  $B_x$  in order to transfer the message to its laboratory. The receivers are allowed to coordinate their actions through classical communication, while quantum communication among them is forbidden.

$B_x$  retrieve the message, each of the other  $(n - 1)$  parties performs a measurement on the Fourier basis  $\{|+\rangle, |-\rangle\}$ ,  $|\pm\rangle := (|0\rangle \pm |1\rangle)/\sqrt{2}$ , collapsing the state of party  $B_x$  to  $|\psi_s\rangle := \alpha|0\rangle + (-1)^s\beta|1\rangle$ , where  $s := \sum_{y \neq x} o_y$  is the sum of the measurement outcomes,  $o_y$  being the measurement outcome obtained by the  $y$ -th party. Finally, the  $n - 1$  parties communicate their outcomes to  $B_x$ , who performs the correction operation  $Z^s$ , with  $Z := |0\rangle\langle 0| - |1\rangle\langle 1|$ . It is easy to see that party  $B_x$  eventually receives the quantum state  $|\psi\rangle$  without any error. All together, this protocol requires 1 qubit of quantum communication from the sender to each receiver.

Now, it may happen that the quantum channels are noisy. For protocols involving a single round of classical communication to the chosen receiver, we show that perfect random-receiver quantum communication is possible only if each the channels  $(\mathcal{C}_i)_{i=1}^n$  can transfer at least one qubit without errors. This result implies that the simple noiseless protocol presented above is optimal in terms of quantum communication. Moreover, the result shows that random-receiver quantum communication cannot take place if the channels  $(\mathcal{C}_i)_{i=1}^n$  are *entanglement-breaking*. As it turns out, this impossibility of random-receiver quantum communication with entanglement-breaking channels holds not only for one-way protocols, but

also for protocols involving arbitrarily many rounds of local operations and classical communication (LOCC).

In contrast with the above observations, in the following we will show that random-receiver quantum communication *can* take place when multiple entanglement-breaking channels are applied in a superposition of alternative orders. Suppose that the quantum communication between the sender and the  $i$ -th receiver takes place through two channels  $\mathcal{A}_i$  and  $\mathcal{B}_i$ , and that the order of application of the two channels is entangled with the state of a control qubit, which we call the *order qubit*. In this scenario, illustrated in Fig. 6.2, we show that perfect random-receiver quantum communication is possible even if all the channels  $(\mathcal{A}_i, \mathcal{B}_i)_{i=1}^n$  are entanglement-breaking, provided that the order qubit is accessible through measurements, and that the outcome of a binary measurement is sent to the chosen receiver. In other words, the indefiniteness of the order enables  $n$ -party random-receiver quantum communication using only entanglement-breaking channels and one bit of classical communication to one of the parties.

Remarkably, this phenomenon cannot be reproduced in a scenario where the order of the channels is definite and the sender can send quantum data to one of the parties, as illustrated in Fig. 6.3. In other words, the access to the qubit that determines the order is a more powerful resource than the noiseless transmission of quantum data from the sender to one of the parties. In fact, we prove an even stronger result: classical communication of the measurement outcomes on the order qubit is a more powerful resource than noiseless quantum communication to  $n - 1$  parties. Achieving random-receiver quantum communication in the scenario of Fig. 6.3 requires at least one qubit of noiseless quantum communication to each of the  $n$  parties.

Our results show that the order qubit can unlock quantum communication to a randomly chosen receiver. The unlocking takes place thanks to the correlations between the order qubits and the output of the noisy channels connecting the sender to the receivers. In contrast, any noiseless quantum communication channel from the sender to a given receiver does not establish correlations with the output of the other receivers. As a consequence, the only way to achieve random-receiver quantum communication through the addition of noiseless communication is to have one noiseless communication channel for each of the  $n$  receivers.

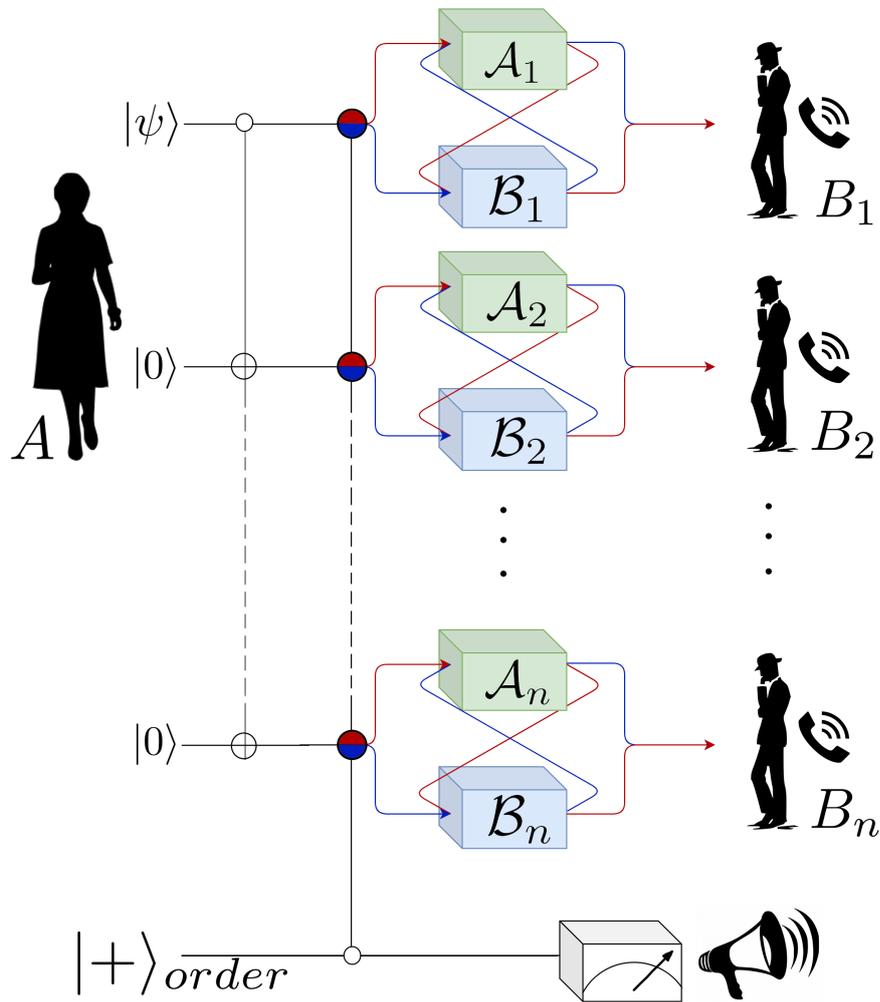


Figure 6.2: **Random-receiver quantum communication through noisy channels.** Perfect protocol with entanglement-breaking channels in a superposition of orders. Alice encodes the unknown qubit state  $\psi$  in a  $n$ -partite GHZ state and sends through the noisy channels. An *order qubit* controls the order in which the subsystems pass through noisy channels, i.e. either  $\mathcal{A}_i$  before  $\mathcal{B}_i$ , or  $\mathcal{B}_i$  before  $\mathcal{A}_i$ , for all  $i \in \{1, \dots, n\}$ . When the order qubit is prepared a superposition of the two classical states  $|0\rangle$  and  $|1\rangle$ , the channels are executed in a coherent superposition of these two orderings. The goal of the protocol is to transfer a generic state  $|\psi\rangle$  from the sender to one of the  $n$  receivers (in this picture, say, receiver  $B_1$ ). To achieve this goal, local measurements are performed on the order qubit and on all the qubits other than the qubit of the designated receiver  $B_1$ . The measurement outcomes are then communicated to  $B_1$ , who performs a correction operation that retrieves Alice's message without error.

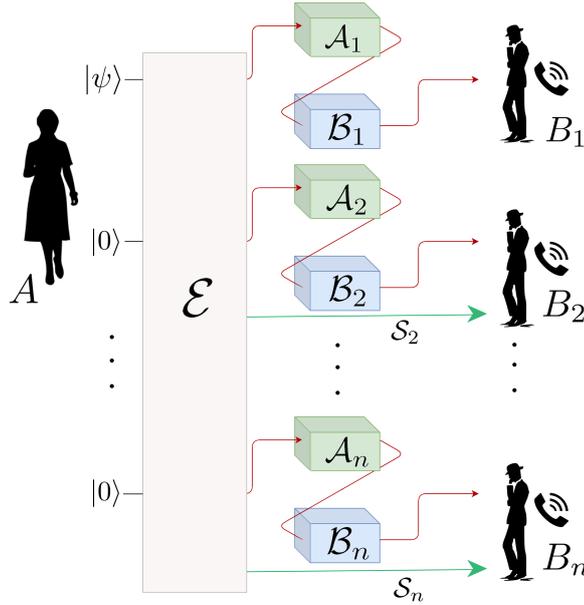


Figure 6.3: **Random-receiver quantum communication with  $k < n$  noiseless side-channels.** Alice communicates to the  $n$  receivers through entanglement breaking channels in a fixed order, with the assistance of  $k < n$  noiseless qubit channels, transferring quantum information from her lab to the labs of  $k$  receivers. In the figure,  $k = n - 1$  and the  $k$  side-channels reach receivers  $B_2, \dots, B_n$ . Despite the assistance of the noiseless side-channels, random-receiver quantum communication cannot be accomplished when the designated receiver is  $B_1$ .

### 6.2.1 Conditions for random-receiver quantum communication.

We first show that the noiseless protocol provided in the introduction is optimal among one-way protocols, that is, protocols consisting of a single round of classical communication to the chosen receiver. For a generic quantum channel  $\mathcal{C}_L$  of a generic quantum system  $L$ , we say that  $\mathcal{C}$  can transmit a  $d$ -dimensional quantum system in a one-way protocol if there exists an encoding channel  $\mathcal{E}_{LR} : L(\mathcal{H}_S) \rightarrow L(\mathcal{H}_L \otimes \mathcal{H}_R)$ , a measurement  $(P_j)_j$  on system  $R$ , and a set of local operations  $(\mathcal{D}_j)_j$  on system  $L$ , such that  $\sum_j \text{Tr}_L[(\mathcal{D}_j \mathcal{C}_L \otimes P_j) \mathcal{E}_{LR}(\rho)] = \rho$  for every state  $\rho \in \text{St}(\mathbb{C}^d)$ .

**Proposition 1:** *If channel  $\mathcal{C}_L$  can transfer the state of a  $d$ -dimensional quantum system in a one-way protocol, then channel  $\mathcal{C}_L$  has quantum capacity of at least  $\log d$ .*

**Proof.** Defining  $\mathcal{E}_j := \text{Tr}_L[(\mathcal{I}_R \otimes P_j) \mathcal{E}_{LR}]$ , we obtain the equivalent condition  $\sum_j \mathcal{D}_j \mathcal{C}_R \mathcal{E}_j = \mathcal{I}_d$ . This condition is satisfied if and only if each term in the sum is proportional to the identity map, namely  $\mathcal{D}_j \mathcal{C}_R \mathcal{E}_j = p_j \mathcal{I}_S$  for some probability distribution  $(p_j)_j$ . Since  $\mathcal{F}_j$  and  $\mathcal{C}_L$  are trace-preserving, this condition implies that  $\mathcal{E}'_j := \mathcal{E}_j / p_j$  is trace-preserving. Since

the condition  $\mathcal{D}_j \mathcal{C}_R \mathcal{E}'_j = \mathcal{I}_d$  holds, the channel  $\mathcal{C}_R$  permits a perfect transmission of a  $d$ -dimensional system, and therefore has a quantum capacity of at least  $\log d$  qubits.

**Theorem 1:** *Every one-way protocol for random-receiver communication of a  $d$ -dimensional quantum message requires each of the channels  $(\mathcal{C}_i)_{i=1}^n$  to have a quantum capacity of at least  $\log d$  qubits.*

**Proof.** Suppose that there exists a one-way protocol for random-receiver quantum communication using channels  $(\mathcal{C}_i)_{i=1}^n$ , and that the protocol can successfully transfer a  $d$ -dimensional quantum system to any of the receivers  $(B_i)_{i=1}^n$ . Let  $\mathcal{E} : L(\mathbb{C}^d) \rightarrow L(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n)$  be the encoding channel used in the protocol. For every  $x \in \{1, \dots, n\}$ , let  $(M_j^{(x)})$  be the measurement performed by the  $(n-1)$  parties other than party  $x$ , and let  $(\mathcal{B}_j^{(x)})$  be the conditional operations performed by party  $x$ . We can then regard systems  $B_x$  and  $\bigotimes_{y \neq x} B_y$  as systems  $L$  and  $R$  in Proposition 1, with encoding channel  $\mathcal{E}_{LR} := (\mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_n) \mathcal{E}$  with  $\mathcal{E}_y := \mathcal{C}_y$  for  $y \neq x$  and  $\mathcal{E}_x := \mathcal{I}_x$ . Applying Proposition 1, we then obtain that channel  $\mathcal{C}_x$  must have a capacity of at least  $\log d$  qubits. Since  $x$  is an arbitrary number in  $\{1, \dots, n\}$ , every channel in the set  $(\mathcal{C}_i)_{i=1}^n$  must have a capacity of at least  $\log d$ .

In particular, Theorem 1 implies that random-receiver quantum communication cannot take place when some of the channels  $(\mathcal{C}_i)_{i=1}^n$  are entanglement-breaking. We recall that entanglement-breaking channels are of the measure-and-prepare form  $\mathcal{C}(\rho) = \sum_j \text{Tr}[M_j \rho] \rho_j$ , where  $(M_j)$  is a quantum measurement and  $\{\rho_j\}$  is a set of output states [235]. Entanglement-breaking channels are the prototype of channels with zero quantum capacity, and therefore they cannot achieve random-receiver quantum communication.

Now, we will focus on the scenario where *all* channels are entanglement-breaking, and ask which additional resources should be added in order to enable random-receiver quantum communication. In the basic model of Fig. 6.1, we replace each entanglement-breaking channel  $\mathcal{C}_i$  with a new channel  $\mathcal{C}_i \otimes \mathcal{S}_i$ , where  $\mathcal{S}_i$  is an additional channel from the sender to the  $i$ -th receiver. For simplicity, we assume that each side-channel  $\mathcal{S}_i$  acts on a quantum system of dimension  $d$ , equal to the dimension of the quantum message. In this setting, we prove that random-receiver communication is possible if and only if each side-channel is noiseless. The proof uses a generalization of Proposition 1 to arbitrary separable protocols. For a generic quantum channel  $\mathcal{S}_L$  on a generic quantum system  $L$ , we say that  $\mathcal{C}$  can transmit a  $d$ -dimensional quantum system in an separable protocol if there exists a system  $R$ , an encod-

ing channel  $\mathcal{E}_{LR} : L(\mathcal{H}_S) \rightarrow L(\mathcal{H}_L \otimes \mathcal{H}_R)$ , and a separable channel  $\mathcal{D} = \sum_j \mathcal{L}_j \otimes \mathcal{R}_j$  where  $\mathcal{L}_j : L(\mathcal{H}_L) \rightarrow L(\mathbb{C}^d)$  and  $\mathcal{R}_j : L(\mathcal{H}_R) \rightarrow \mathbb{C}$  are completely positive maps for every  $j$ , such that

$$\mathcal{D}(\mathcal{S}_L \otimes \mathcal{I}_R)\mathcal{E} = \mathcal{I}_d, \quad (6.1)$$

where  $\mathcal{I}_R$  is the identity channel on system  $R$ .

**Proposition 2:** *If the input of channel  $\mathcal{S}_L$  is a  $d$ -dimensional quantum system and  $\mathcal{S}_L$  can transfer the state of a  $d$ -dimensional quantum system in a separable protocol, then  $\mathcal{S}_L$  is a unitary channel.*

**Proof.** Defining  $\mathcal{E}_j := (\mathcal{I}_L \otimes \mathcal{R}_j)\mathcal{E}_{LR}$ , we can rewrite Eq. (6.1) as  $\sum_j \mathcal{L}_j \mathcal{S}_L \mathcal{E}_j = \mathcal{I}_d$ . This condition is satisfied if and only if each term in the sum is proportional to the identity map, namely  $\mathcal{D}_j \mathcal{C}_R \mathcal{E}_j = p_j \mathcal{I}_S$  for some probability distribution  $(p_j)_j$ . Using Kraus representations for the maps  $\mathcal{L}_j$ ,  $\mathcal{S}_L$  and  $\mathcal{E}_j$ , we obtain the condition  $F_{jk} S_l E_{jm} = \lambda_{jklm} I_d$  for suitable coefficients  $\lambda_{jklm}$  satisfying the normalization condition  $\sum_{j,k,l,m} |\lambda_{jklm}|^2 = 1$ . Due to the normalization condition, there must exist values of the indices  $(j, k, l, m)$  such that  $\lambda_{jklm} \neq 0$ . For these values, the condition  $F_{jk} S_l E_{jm} = \lambda_{jklm} I_d$  implies that the operator  $F_{jk}$  is invertible, and that one has  $S_l E_{jm} = \lambda_{jklm} F_{jk}^{-1}$ . Multiplying by  $F_{jk}$  on both sides of the equation, we then obtain

$$S_l E_{jm} F_{jk} = \lambda_{jklm} I_d. \quad (6.2)$$

Now, define the completely positive map  $\tilde{\mathcal{E}}$  by  $\tilde{\mathcal{E}}(\rho) := \sum_{j,k,m} E_{jm} F_{jk} \rho F_{jk}^\dagger E_{jm}^\dagger$ . Eq. (6.2) implies the relation  $\mathcal{S}_L \tilde{\mathcal{E}} = \mathcal{I}_d$ . Since  $\mathcal{S}_L$  and  $\mathcal{I}_d$  are trace-preserving, also  $\tilde{\mathcal{E}}$  must be trace-preserving. Hence,  $\tilde{\mathcal{E}}$  is an invertible quantum channel and  $\mathcal{S}_L$  is its inverse. Since the input and output systems of these two channels have the same dimension, this means that both channels must be unitary.

**Theorem 2:** *Random-receiver quantum communication with entanglement-breaking channels  $(\mathcal{C}_i)_{i=1}^n$  and side-channels  $(\mathcal{S}_i)_{i=1}^n$  is possible if and only if all side-channels are noiseless.*

**Proof.** Suppose that there exists a general LOCC protocol for random-receiver quantum communication using channels  $(\mathcal{C}_i)_{i=1}^n$  and side-channels  $(\mathcal{S}_i)_{i=1}^n$  acting on  $d$ -dimensional quantum systems. Let  $\mathcal{E} : L(\mathbb{C}^d) \rightarrow L(\mathcal{H}'_1 \otimes \cdots \otimes \mathcal{H}'_n)$  be the encoding channel used in the

protocol, with  $\mathcal{H}'_i := \mathcal{H}_{B_i} \otimes \mathcal{H}_{S_i}$ , where  $\mathcal{H}_i$  is the output of channel  $\mathcal{C}_i$  and  $S_i = \mathbb{C}^d$  is the output of the side-channel  $\mathcal{S}_i$ .

For every  $x \in \{1, \dots, n\}$ , we can regard systems  $S_x$  and  $B_x \otimes \left[ \bigotimes_{y \neq x} (B_y \otimes S_y) \right]$  as systems  $L$  and  $R$  in Proposition 2, with channel  $\mathcal{S}_L := \mathcal{S}_x$  and encoding channel  $\mathcal{E}_{LR} := (\mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_n)\mathcal{E}$  with  $\mathcal{E}_y := \mathcal{C}_y \otimes \mathcal{S}_j$  for  $y \neq x$  and  $\mathcal{E}_x := \mathcal{C}_{B_x} \otimes \mathcal{I}_{S_x}$ . The original LOCC protocol can be regarded as a special case of separable protocol with respect to the bipartition  $(L, R)$ . Applying Proposition 1, we then obtain that channel  $\mathcal{S}_x$  must be unitary. Since  $x$  is an arbitrary element of  $\{1, \dots, n\}$ , every channel in the set  $(\mathcal{S}_i)_{i=1}^n$  must be unitary.

In particular, Theorem 2 shows that random-receiver quantum communication with entanglement breaking channels is impossible even if one provides noiseless side-channels to  $k < n$  receivers.

### 6.3 RANDOM-RECEIVER QUANTUM COMMUNICATION THROUGH THE QUANTUM SWITCH

Let  $\mathcal{A} := \bigotimes_{i=1}^n \mathcal{A}_i$  and  $\mathcal{B} := \bigotimes_{i=1}^n \mathcal{B}_i$  be two quantum channels, describing the noise experienced by the data transmitted by a sender to  $n$  receivers. The action of the channels  $\mathcal{A}$  and  $\mathcal{B}$  in a superposition of two alternative orders is described by the *quantum SWITCH* (as defined in chapter 2), a higher-order map that transforms the pair of channels  $(\mathcal{A}, \mathcal{B})$  into a new quantum channels  $\mathcal{S}(\mathcal{A}, \mathcal{B})$ , involving a control qubit that determines the order of application of channels  $\mathcal{A}$  and  $\mathcal{B}$ . In its simplest version, the *quantum SWITCH* produces the channel  $\mathcal{S}(\mathcal{A}, \mathcal{B})$  with Kraus operators

$$W_{jk} := A_j B_k \otimes |0\rangle\langle 0| + B_k A_j \otimes |1\rangle\langle 1|, \quad (6.3)$$

where  $\{A_j\}$  and  $\{B_k\}$  are Kraus representations for channels  $\mathcal{A}$  and  $\mathcal{B}$ , respectively. It is easy to verify that the definition of channel  $\mathcal{S}(\mathcal{A}, \mathcal{B})$  is independent of the choice of Kraus representations. When the order qubit is initialized in the state  $\omega$  we use the shorthand  $\mathcal{S}_\omega(\mathcal{A}, \mathcal{B})(\rho) := \mathcal{S}(\mathcal{A}, \mathcal{B})(\rho \otimes \omega)$ , and we call  $\mathcal{S}_\omega(\mathcal{A}, \mathcal{B})$  the *switched channel*.

### 6.3.1 Switching products of Pauli channels

Let  $\mathcal{A}$  and  $\mathcal{B}$  are products of Pauli channels, given by  $\mathcal{A} \equiv \{\sqrt{p_0}\mathbb{I}, \sqrt{p_1}X, \sqrt{p_2}Y, \sqrt{p_3}Z\}$  and  $\mathcal{B} \equiv \{\sqrt{q_0}\mathbb{I}, \sqrt{q_1}X, \sqrt{q_2}Y, \sqrt{q_3}Z\}$  with  $\sum_{i=0}^3 p_i^2 = \sum_{i=0}^3 q_i^2 = 1$ . The switched channel has the simple expression

$$\mathcal{S}_\omega(\mathcal{A}, \mathcal{B})[\rho] = p_+ \mathcal{C}_+(\rho) \otimes \omega_+ + p_- \mathcal{C}_-(\rho) \otimes \omega_-, \quad (6.4)$$

where  $(p_+, p_-)$  are two probabilities,  $\omega_+ := \omega$  and  $\omega_- := Z\omega Z$  are states of the order qubit, and  $(\mathcal{C}_+(\rho), \mathcal{C}_-(\rho))$  are two quantum channels given by,

$$\begin{aligned} \mathcal{C}_+(\rho) = & \left( \sum_{i=0}^3 p_i^2 q_i^2 + p_0^2 (1 - q_0^2) + p_1^2 (1 - q_1^2) + p_2^2 (1 - q_2^2) + p_3^2 (1 - q_3^2) \right) \rho \\ & + 2q_0 \sum_{i=0}^3 p_i^2 (q_1(\mathbb{I} \otimes X)\rho(\mathbb{I} \otimes X) + q_2(\mathbb{I} \otimes Y)\rho(\mathbb{I} \otimes Y) + q_3(\mathbb{I} \otimes Z)\rho(\mathbb{I} \otimes Z)) \\ & + 2p_0 \sum_{i=0}^3 q_i^2 (p_1(X \otimes \mathbb{I})\rho(X \otimes \mathbb{I}) + p_2(Y \otimes \mathbb{I})\rho(Y \otimes \mathbb{I}) + p_3(Z \otimes \mathbb{I})\rho(\mathbb{I} \otimes Z)) \\ & + 4(p_0 p_1 q_0 q_1 + p_2 p_3 q_2 q_3)(X \otimes X)\rho(X \otimes X) + 4(p_0 p_2 q_0 q_2 + p_1 p_3 q_1 q_3)(Y \otimes Y)\rho(Y \otimes Y) \\ & + 4(p_0 p_3 q_0 q_3 + p_2 p_1 q_2 q_1)(Z \otimes Z)\rho(Z \otimes Z) + 4(p_0 p_1 q_0 q_2 + p_2 p_3 q_1 q_3)(X \otimes Y)\rho(X \otimes Y) \\ & + 4(p_0 p_1 q_0 q_3 + p_2 p_3 q_1 q_2)(X \otimes Z)\rho(X \otimes Z) + 4(p_0 p_2 q_0 q_1 + p_1 p_3 q_2 q_3)(Y \otimes X)\rho(Y \otimes X) \\ & + 4(p_0 p_2 q_0 q_3 + p_1 p_3 q_1 q_2)(Y \otimes Z)\rho(Y \otimes Z) + 4(p_0 p_3 q_0 q_1 + p_2 p_1 q_2 q_3)(Z \otimes X)\rho(Z \otimes X) \\ & + 4(p_0 p_3 q_0 q_2 + p_1 p_2 q_1 q_3)(Z \otimes Y)\rho(Z \otimes Y); \end{aligned} \quad (6.5)$$

$$\begin{aligned} \mathcal{C}_-(\rho) = & 2 \sum_{i=0}^3 p_i^2 (q_1 q_2 (\mathbb{I} \otimes Z)\rho(\mathbb{I} \otimes Z) + q_2 q_3 (\mathbb{I} \otimes X)\rho(\mathbb{I} \otimes X) + q_3 q_1 (\mathbb{I} \otimes Y)\rho(\mathbb{I} \otimes Y)) \\ & + 2 \sum_{i=0}^3 q_i^2 (p_1 p_2 (Z \otimes \mathbb{I})\rho(Z \otimes \mathbb{I}) + p_2 p_3 (X \otimes \mathbb{I})\rho(X \otimes \mathbb{I}) + p_3 p_1 (Y \otimes \mathbb{I})\rho(Y \otimes \mathbb{I})) \\ & + 4(p_2 p_3 q_0 q_1 + p_0 p_1 q_2 q_3)(X \otimes X)\rho(X \otimes X) + 4(p_1 p_3 q_0 q_2 + p_0 p_2 q_1 q_3)(Y \otimes Y)\rho(Y \otimes Y) \\ & + 4(p_1 p_2 q_0 q_3 + p_0 p_3 q_1 q_2)(Z \otimes Z)\rho(Z \otimes Z) + 4(p_0 p_1 q_1 q_2 + p_2 p_3 q_0 q_3)(X \otimes Z)\rho(X \otimes Z) \\ & + 4(p_0 p_1 q_1 q_3 + p_2 p_3 q_0 q_2)(X \otimes Y)\rho(X \otimes Y) + 4(p_1 p_3 q_0 q_1 + p_0 p_2 q_2 q_3)(Y \otimes X)\rho(Y \otimes X) \\ & + 4(p_1 p_3 q_0 q_3 + p_0 p_2 q_1 q_2)(Y \otimes Z)\rho(Y \otimes Z) + 4(p_1 p_2 q_0 q_1 + p_0 p_3 q_2 q_3)(Z \otimes X)\rho(Z \otimes X) \\ & + 4(p_1 p_2 q_0 q_2 + p_0 p_3 q_1 q_3)(Z \otimes Y)\rho(Z \otimes Y). \end{aligned} \quad (6.6)$$

In the following, we will focus on the case where all channels  $(\mathcal{A}_i)_{i=1}^n$  and  $(\mathcal{B}_i)_{i=1}^n$  are equal to the Pauli channel  $\mathcal{N}_{XY}$ , defined by  $\mathcal{N}_{XY}(\rho) = 1/2(X\rho X + Y\rho Y)$ . This channel is entanglement-breaking and therefore cannot directly transmit quantum information. However, we will see that the use of this channel in a superposition of orders achieves perfect quantum communication to a randomly chosen receiver.

### 6.3.2 Random-receiver quantum communication for 2 receivers

For simplicity, let us first illustrate the idea for  $n = 2$ . First, the sender encodes the message  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$  into the state  $|\psi_2\rangle := \alpha|00\rangle + \beta|11\rangle$ , as in the noiseless protocol. Then, the sender sends the two qubits to receivers 1 and 2, using the channels  $\mathcal{A} = \mathcal{N}_{XY} \otimes \mathcal{N}_{XY}$  and  $\mathcal{B} = \mathcal{N}_{XY} \otimes \mathcal{N}_{XY}$  in a superposition of orders. When the order qubit is initialized in the state  $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ , we get the expression of  $\mathcal{C}_\pm$  by putting  $p_0 = p_3 = q_0 = q_3 = 0$  and  $p_1 = p_2 = q_1 = q_2 = \frac{1}{\sqrt{2}}$  in Eqs.(6.5-6.6),

$$\begin{aligned}\mathcal{C}_+(\rho) &= \frac{\rho + (Z \otimes Z)\rho(Z \otimes Z)}{2} \\ \mathcal{C}_-(\rho) &= \frac{(I \otimes Z)\rho(I \otimes Z) + (Z \otimes I)\rho(Z \otimes I)}{2},\end{aligned}\tag{6.7}$$

and the probabilities  $p_\pm$  are both equal to 1/2. The output states of the order qubit are either  $\omega_+ = |+\rangle\langle+|$  or  $\omega_- = |-\rangle\langle-|$ , with  $|-\rangle := (|0\rangle - |1\rangle)/\sqrt{2}$ . Since these two states are orthogonal, a measurement on the order qubit postselect one of the two channels  $\mathcal{C}_+$  or  $\mathcal{C}_-$ . Moreover, the channels  $\mathcal{C}_+$  and  $\mathcal{C}_-$  are equivalent under local unitary operations: for example, party 1 can turn channel  $\mathcal{C}_-$  into channel  $\mathcal{C}_+$  by applying the Pauli gate  $Z$  on its qubit. If the outcome of the measurement on the order qubit is shared to the two receivers, they can ensure that their qubits have gone through the channel  $\mathcal{C}_+$ . Now, the pure state  $\rho = |\psi_2\rangle\langle\psi_2|$  is invariant under the action of channel  $\mathcal{C}_+$ , and therefore it reaches the two receivers without any error. Hence, the two receivers end up with two qubits in the same state as in the noiseless protocol, and can achieve random-receiver quantum communication. Summarizing, classical communication of the outcome of a measurement on the order qubit

enables perfect random-receiver quantum communication.

The above protocol can be generalized from  $n = 2$  to arbitrary numbers of receivers. Below, we discuss about the protocol for  $n$  (where  $n > 0$  can be arbitrarily large) number of receivers.

### 6.3.3 Random-receiver quantum communication for $n$ receivers

When  $\mathcal{A} = \mathcal{B} := \otimes_{k=1}^n \mathcal{N}_{XY}$  the switched quantum distribution channel from Alice to  $n$  Bobs read as,

$$\begin{aligned} \mathcal{S}_{\omega_c}^{(n)}[\rho_{B_1..B_n}] &:= \sum_{i_1, j_1, \dots, i_n, j_n=0}^1 G_{i_1 j_1 .. i_n j_n}(\rho_{B_1..B_n} \otimes \omega_c) G_{i_1 j_1 .. i_n j_n}^\dagger; \\ \text{where, } G_{i_1 j_1 .. i_n j_n} &:= \otimes_{k=1}^n E_{i_k} F_{j_k} \otimes |0\rangle_c \langle 0| + \otimes_{k=1}^n F_{j_k} E_{i_k} \otimes |1\rangle_c \langle 1|; \\ \text{with, } E_0 = F_0 &= \frac{1}{\sqrt{2}} X, \text{ and, } E_1 = F_1 = \frac{1}{\sqrt{2}} Y. \end{aligned} \quad (6.8)$$

Consider now an individual term in the summation of the right hand side of Eq.(6.8). The sign of the coherence term of the order qubit will be flipped, i.e.,  $\omega_c \rightarrow Z\omega_c Z$  if  $\bigoplus_{k=1}^n (i_k \oplus j_k) = 1$ , and whenever  $\bigoplus_{k=1}^n (i_k \oplus j_k) = 0$  it will remain invariant, i.e.,  $\omega_c \rightarrow \omega_c$ . Furthermore we will use the facts that  $XX = YY = \mathbb{I}$  and  $XY = iZ$  and  $YX = -iZ$  in our following analysis.

**Case-I:** Order bit invariant terms [ $\bigoplus_{k=1}^n (i_k \oplus j_k) = 0$ ]. In this case we have the following terms:

- (i)  $i_k = j_k, \forall k$ ; which will result the term  $G_{i_1 j_1 .. i_n j_n}$  of the form  $\mathbb{I}_{B_1..B_n} \otimes (|0\rangle_c \langle 0| + |1\rangle_c \langle 1|)$  at right hand side of Eq.(6.8).
- (ii) For even number of cases (say  $2m$ ) the indices  $i_k$ 's are different than the corresponding  $j_k$ 's and for the other cases they are equal. For a given  $m \in \{0, 1, \dots, \lfloor \frac{n}{2} \rfloor\}$ , the terms  $G_{i_1 j_1 .. i_n j_n}$  will be  $(-1)^m (Z_1, \dots, Z_{2m}, \mathbb{I}_{2m+1..n}) \otimes (|0\rangle_c \langle 0| + |1\rangle_c \langle 1|)$ , where  $(Z_1, \dots, Z_p, \mathbb{I}_{p+1..q})$  denotes term with  $Z$  acting on  $p$  among  $q$  numbers of state and identity acting on rest. Number of such terms be  $\binom{q}{p = \frac{q!}{p!(q-p)!}}$ .

**Case-II:** Order bit flipped terms  $[\bigoplus_{k=1}^n (i_k \oplus j_k) = 1]$ . In this case we have the following terms:

- (i) For odd number of cases (say  $2m + 1$ ) the indices  $i_k$ 's and the corresponding  $j_k$ 's are  $i_k = \bar{j}_k = 0$  and for the other cases they are equal. For a given  $m \in \{0, 1, \dots, \lfloor \frac{n-1}{2} \rfloor\}$  this will result terms  $G_{i_1 j_1 \dots i_n j_n}$  of the form,  $i \times (-1)^m (Z_1, \dots, Z_{2m+1}, \mathbb{I}_{2m+2}, \dots, \mathbb{I}_n) \otimes (|0\rangle_c \langle 0| - |1\rangle_c \langle 1|)$ .
- (ii)  $2m + 1$  numbers of  $i_k = \bar{j}_k = 1$  and rests are equal. For a given  $m \in \{0, 1, \dots, \lfloor \frac{n-1}{2} \rfloor\}$  the terms  $G_{i_1 j_1 \dots i_n j_n}$  will be of the form,  
 $-i \times (-1)^m (Z_1, \dots, Z_{2m+1}, \mathbb{I}_{2m+2}, \dots, \mathbb{I}_n) \otimes (|0\rangle_c \langle 0| - |1\rangle_c \langle 1|)$ .

Combining these all together we finally have,

$$\begin{aligned} \mathcal{S}_{\omega_c}^{(n)}[\rho_{B_1 \dots B_n}] &:= \frac{1}{2^n} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} [(Z_1, \dots, Z_{2m}, \mathbb{I}_{2m+1}, \dots, \mathbb{I}_n) \rho_{B_1 B_2 \dots B_n} (Z_1, \dots, Z_{2m}, \mathbb{I}_{2m+1}, \dots, \mathbb{I}_n)] \otimes \omega_c \\ &+ \frac{1}{2^n} \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} [(Z_1, \dots, Z_{2m+1}, \mathbb{I}_{2m+2}, \dots, \mathbb{I}_n) \rho_{B_1 B_2 \dots B_n} (Z_1, \dots, Z_{2m+1}, \mathbb{I}_{2m+2}, \dots, \mathbb{I}_n)] \otimes Z \omega_c Z. \end{aligned} \quad (6.9)$$

Suppose that initial state of order system is  $\omega_c = |+\rangle\langle +|$ . In that case after the evolution of switched channel if depending on the outcome of Pauli X measurement on order system if one of the receivers apply suitable local unitary correction on his subsystem, then the final outcome state reads as,

$$\frac{1}{2^n} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} [(Z_1, \dots, Z_{2m}, \mathbb{I}_{2m+1}, \dots, \mathbb{I}_n) \rho_{B_1 B_2 \dots B_n} (Z_1, \dots, Z_{2m}, \mathbb{I}_{2m+1}, \dots, \mathbb{I}_n)]. \quad (6.10)$$

In the present context the input state is the generalized GHZ state  $|\psi_{B_1 \dots B_n}\rangle = \alpha|0 \dots 0\rangle_{B_1 \dots B_n} + \beta|1 \dots 1\rangle_{B_1 \dots B_n}$  which is invariant under local unitary Z operation by any even number of parties. Thus the state gets distributed perfectly among  $n$  receivers.

To reproduce the qubit information at one of the receivers' lab they will follow a LOCC protocol. Suppose that the qubit state needs to be reproduced at  $i^{th}$  Bob. All other Bobs will perform Pauli X measurement on their respective subsystems and inform the measurement results  $x_k \in \{+1, -1\}$ ,  $\forall k \in \{1, \dots, n\}$  &  $k \neq i$ . Depending on this information  $i^{th}$  Bob will apply Z unitary correction on his part if  $\prod_{k \neq i} x_k$  is  $-1$ , otherwise he does nothing.

The crucial feature of the protocol is that access to a single qubit (the order qubit) is enough to unlock quantum communication to  $n$  independent receivers. This feature cannot be reproduced by adding a qubit side-channel in a causally ordered scenario. In fact, Theorem 2 implies that random-receiver quantum communication is impossible even if one adds any number  $k < n$  of qubit side-channels. In short, the mere access to the order qubit is a more powerful resource than the access to  $(n - 1)$  qubit side-channels.

#### 6.4 INVESTIGATIONS OF OTHER RESOURCES THAT EXHIBIT RANDOM-RECEIVER QUANTUM COMMUNICATION TASK PERFECTLY IN CAUSALLY ORDERED SCENARIO

Quantum communication with the assistance of the *quantum SWITCH* is similar to quantum communication with classical assistance from the environment [236–239]. In both cases, the access to a measurement outcome unlocks some quantum information that would be inaccessible otherwise. The analogy goes even further, because the *quantum SWITCH* of two Pauli channels  $\mathcal{A}$  and  $\mathcal{B}$  is an extension of the quantum channel  $\mathcal{AB}$ , that is, the channel that arises when channels  $\mathcal{A}$  and  $\mathcal{B}$  are applied in cascade in a definite causal order. Precisely, the channel  $\mathcal{AB}$  can be obtained from the switched channel  $\mathcal{S}_\omega(\mathcal{A}, \mathcal{B})$  by discarding the order system. From this point of view, the order qubit is indeed part of the environment of the channel  $\mathcal{AB}$ , and quantum communication with the assistance of the SWITCH is a special case of quantum communication with classical assistance from the environment. The key difference is that, in the case of the *quantum SWITCH*, only a small part of the environment needs to be accessible, while in the other examples of quantum communication with the assistance of environment it is generally assumed that the whole environment be accessible.

Another class of communication protocols that exhibit similarities with the *quantum SWITCH* are the communication protocols using controlled operations before and after the communication channels [240]. Like the *quantum SWITCH*, these protocols use a control qubit, which determines the choice of operations performed on the input and output of the communication channels. The key difference with the *quantum SWITCH* is that such protocols generally transfer information to the control system in a way that bypasses the original channels [241, 242]. In contrast, in all the protocols considered in the literature, the *quantum SWITCH* does

not deposit information into the order qubit. For protocols involving Pauli channels, this feature is evident from Eq. (6.4), where the states  $\omega_{\pm}$  of the order qubit are independent of the message, and so are the probabilities  $p_{\pm}$ .

We observe that, if we allow arbitrary controlled operations before and after the noisy channels, then protocols for random-receiver quantum communication with entanglement-breaking channels can be constructed also in the causally ordered scenario. This is because controlled operations can be used (i) to transfer information directly from the message to the control qubit, bypassing the noisy channels  $\mathcal{A}$  and  $\mathcal{B}$ , and (ii) to generate the generalized GHZ state  $\alpha|0\rangle^{\otimes n} + \beta|1\rangle^{\otimes n}$  from the state of the control qubit, evading the locality restriction that affects the receivers. The example of protocols that achieve random-receiver communication through controlled operations in a definite causal order is presented below.

#### 6.4.1 Protocol for random-receiver quantum communication using arbitrary controlled operations

The following protocol permits random-receiver quantum communication through the channels  $\mathcal{A} = \mathcal{B} = \mathcal{N}_{XY}^{\otimes n}$ . The protocol starts with the message encoded in the state of the first qubit. First, the sender applies a CNOT gate to the message and the control qubit, initialized in the state  $|+\rangle$ . As a result, the message and the control qubit end up in the state  $[|0\rangle(\alpha|0\rangle + \beta|1\rangle) + |1\rangle(\alpha|1\rangle + \beta|0\rangle)]/\sqrt{2}$ . Second, the sender prepares  $n - 1$  qubits in the state  $|0\rangle$ . The sender sends the message and the other  $n - 1$  qubits through the channel  $\mathcal{AB} = (\mathcal{N}_{XY}^2)^{\otimes n}$ , which collapses the overall state a classical mixture of the states  $|0\rangle|0\rangle^{\otimes(n-1)}(\alpha|0\rangle + \beta|1\rangle)$  and  $|1\rangle|0\rangle^{\otimes(n-1)}(\alpha|1\rangle + \beta|0\rangle)$ . Third, CNOT gates are applied to the control qubit and to the additional  $n - 1$  qubits, producing either the state  $|0\rangle(\alpha|0\rangle^{\otimes n} + \beta|1\rangle^{\otimes n})$  or the state  $|1\rangle(\alpha|1\rangle^{\otimes n} + \beta|0\rangle^{\otimes n})$ . Finally, the first receiver measures the first qubit in the computational basis, and, if the outcome is  $\mathbf{1}$ , all the other receivers perform the bit flip operation  $X = |0\rangle\langle 1| + |1\rangle\langle 0|$  on the remaining qubits and on the control qubit. In this way, the remaining qubits end up in the generalized GHZ state  $\alpha|0\rangle^{\otimes n} + \beta|1\rangle^{\otimes n}$ , and random-receiver quantum communication can be achieved as in the noiseless protocol presented in the introduction.

Note that this protocol uses the control qubit to bypass the noisy channels  $\mathcal{A}$  and  $\mathcal{B}$ , as one can see from the fact that, after the channels  $\mathcal{A}$  and  $\mathcal{B}$  have acted, all the information about the message is on the control qubit. In addition, the protocol freely generates entanglement between the  $n$  receivers after the channels  $\mathcal{A}$  and  $\mathcal{B}$  have acted. The entanglement generation is achieved by the  $n - 1$  CNOT gates applied in the last step of the protocol. In general, these CNOTs cannot be implemented by the receivers, due to their spatial separation. Hence, they must be regarded as performed by a third party other than the receivers.

Note that the presence of entangling operations between the control and each receiver is essential in the above protocol. More generally, entangling operations are necessary in any protocol that achieves random-receiver quantum communication through entanglement-breaking channels in a definite order. Any such protocol needs to bypass the entanglement-breaking channels by encoding quantum information in the control qubit. An equivalent condition can be obtained by introducing an additional reference system at the receiver's end: in order to achieve perfect quantum communication, the protocol must transform a maximally entangled state of the input and the reference into a maximally entangled state of the control and the reference. After the action of an entanglement-breaking channel, the control and the reference have no correlation with the qubits at the receivers' locations. Hence, no quantum information can be transferred back from the control to the receivers without the use of entangling operations.

#### 6.4.2 No GHZ state generation with controlled SWAP operations

Arbitrary controlled operations appear to be a too broad set for the problem of random-receiver quantum communication, in that they allow a complete transfer of information to the control qubit and violate the locality restrictions among the receivers. On the other hand, The *quantum SWITCH* can be regarded as a controlled SWAP operation *in time*: it swaps the order of quantum systems appearing in a given time sequence, putting the inputs/outputs of channel  $\mathcal{A}$  either before or after the inputs/outputs of channel  $\mathcal{B}$ . One may then ask if controlled SWAP operations *in space* can reproduce the same features when the channels  $\mathcal{A}$  and  $\mathcal{B}$  are arranged in a fixed sequential order, say with  $\mathcal{A}$  acting before  $\mathcal{B}$ .

Here we show that the answer is negative, in the following sense: suppose that the quantum channels  $\mathcal{A}$  and  $\mathcal{B}$  are placed in a fixed order, and that their inputs and outputs undergo controlled permutations, as in Fig. 6.4. For odd  $n$ , we show that, no matter which controlled permutations are chosen, one of the receivers will remain in a fixed state, independent of the quantum message from the sender.

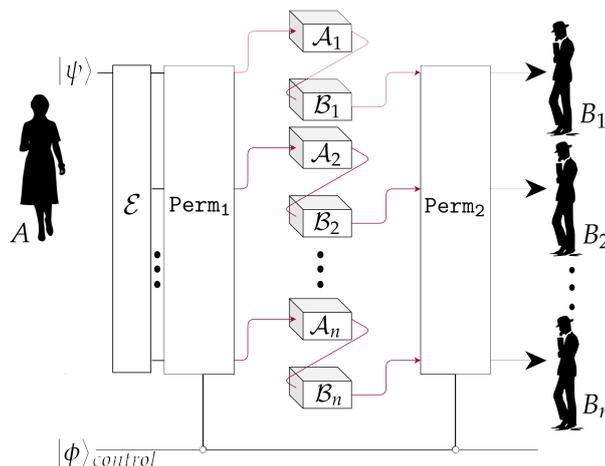


Figure 6.4: Random-receiver quantum communication task with controlled SWAP operations *in space*.

The argument is simple. The output of channel  $\mathcal{N}_{XY}^{\otimes n}$  is a mixture of states in the computational basis, of the form  $\otimes_{j=1}^n |b_j\rangle$ , with  $b_j \in \{0,1\}$  for every  $j$ . Perfect quantum communication through the entanglement-breaking channel is possible only if, for every state appearing in the mixture, the initial message has been perfectly transferred to the control qubit.

Let us focus on one specific state in the mixture, say  $|\mathbf{b}\rangle = \otimes_{j=1}^n |b_j\rangle$ , and let us denote by  $|\psi'\rangle = \alpha'|0\rangle + \beta'|1\rangle$  the state of the control qubit conditional to the state  $\otimes_{j=1}^n |b_j\rangle$ . The state  $|\mathbf{b}\rangle|\psi'\rangle$  then undergoes a controlled operation  $W$ , becoming the state  $W|\mathbf{b}\rangle|\psi'\rangle = \alpha' \otimes_{j=1}^n |b_{\pi(j)}\rangle \otimes |0\rangle + \beta' \otimes_{j=1}^n |b_{\sigma(j)}\rangle \otimes |1\rangle$ , where  $\pi$  and  $\sigma$  are two permutations. Equivalently, the state can be rewritten as  $W|\mathbf{b}\rangle|\psi'\rangle = \alpha' \otimes_{j=1}^n |b_j\rangle \otimes |0\rangle + \beta' \otimes_{j=1}^n |b_{\tau(j)}\rangle \otimes |1\rangle$  for some suitable permutation  $\tau$ . When  $n$  is odd, there exists at least one value of  $j$  such that  $b_j = \tau(b_j)$ . Hence, the  $j$ -th system ends up in the state  $|b_j\rangle$ , which has no dependence on the coefficients  $\alpha'$  and  $\beta'$ , and therefore on the initial message.

In summary, the  $n$  output systems and the control end up in a mixture of pure states, one of which is a product between a fixed state of system  $j$  and the remaining systems. Hence,

the quantum message cannot be transferred perfectly to the  $j$ -th receiver, since with some non-zero probability, the state of the  $j$ -th system will be independent of the message.

The possibility of random-receiver quantum communication through controlled operations in a definite order can be interpreted in two ways. On the one hand, controlled operations can generate entanglement among the  $n$  receivers, and therefore appear to be too powerful to be interesting in the problem of random-receiver quantum communication, where locality in space is an essential feature of the problem. On the other hand, controlled operations have some similarity with the *quantum SWITCH*, which can be regarded as a controlled SWAP operation *in time*. Controlled SWAP operations are a special subset of the set of all controlled operations, and one may wonder whether this special subset can reproduce the features of the *quantum SWITCH*. Interestingly, the answer is negative. We have shown that no controlled routing of the inputs and outputs of channels  $\mathcal{A} = \mathcal{B} = \mathcal{N}_{XY}^{\otimes n}$  permits random-receiver quantum communication for odd  $n$ .

## 6.5 PHOTONIC SIMULATION OF THE RANDOM-RECEIVER QUANTUM COMMUNICATION TASK

*Quantum SWITCH* has recently been simulated in various photonic setups [34, 230–232]. For instance, in the scheme of Ref.[34] photon’s transverse spatial mode behaves as the target system evolving under two quantum operations whose relative order is controlled by photon’s polarization degrees of freedom (DOF). For implementing random-receiver quantum communication through *quantum SWITCH* in photonic setup we require more than two DOFs to be considered at a time with one of them playing the role of order system. In the present context, it is assumed that the sender possesses advanced optical devices that allow her to apply any joint (entangled) quantum operation on multiple DOFs of the photon, whereas the receiver can address each DOF individually. This assumption effectively mimics the scenario of random-receiver quantum communication with different DOFs playing the role of different spatially separated Bobs. In which DOF the quantum information has to be reproduced is decided at a later time after the DOFs evolve through noisy processes. Multiple DOFs of photon, such as polarization, spatial-mode, orbit-angular-momentum, time-bin and

frequency have already been addressed simultaneously in different photonic experiments [243–246]. The proposed random-receiver quantum communication task thus welcomes an inquisitive conglomeration of presently available quantum optical devices to demonstrate a novel information theoretic advantage of indefinite causal order.

## 6.6 SUMMARY AND OUTLOOK

Coherent control of orders/paths of quantum process has gained much of recent interest as it finds useful applications in quantum communication tasks [30, 36–39, 65, 222–229, 247]. To what extent these advantages are specific to superpositions of causal orders, rather than being generic to other forms of coherent superpositions of communication protocols, is currently a matter of debate [240–242]. In particular, the advantage of coherent control of orders in time over that of paths in space is achieved under the distinct role of external and internal degrees of freedoms in communication task. In this regard the present work is quite important. Here, a novel generalization of quantum communication task has been introduced and the advantage of indefinite ordering of quantum processes has been established over coherently controlled processes with fixed order. Importantly, this advantage implies that access to a qubit system, controlling the order of quantum processes, is a more powerful resource than  $(n - 1)$  qubit side-channels for any natural number  $n > 1$ .

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## CONCLUSIONS AND FUTURE DIRECTIONS

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In this chapter, several salient features of quantum information theoretic resources, presented through out this dissertation have been summarized with speculation on some future directions. As mentioned in the introduction, entanglement and various other quantum correlations are shown to be useful resources in various information theoretic tasks which otherwise are not possible in classical theory. However, before utilizing these resources in any information processing task, at first one needs to detect their signature, which is operationally a hard task. In the present thesis, we proposed some detection schemes for several information theoretic resources, which can be easily performed in real experiments. Among them, we presented detection schemes for EPR-steering, non-Markovianity and sequentially shared genuine tripartite entanglement. Apart from these, in this dissertation an application of indefinite causal order has also been proposed by introducing a new communication game called *random-receiver quantum communication*. The game establishes the advantage of indefinite ordering of quantum processes over coherently controlled processes with fixed order. Let us now briefly summarize the main results presented in this thesis:

In Chapter 3, a new steering criterion has been derived using the sum-uncertainty relation. Since, sum-uncertainty relation provides a tighter uncertainty bound compared to the Heisenberg uncertainty relation, the sum-steering inequality is expected to yield a tighter steering relation compared to the Reid-inequality which is based on the Heisenberg uncertainty relations. Our proposed steering criterion is able to replicate the optimal steering range of Werner states for the case of bipartite systems with two measurement settings and two outcomes, and thus, establishes its superiority over the entropic steering criterion. In

the case of continuous variable system, the sum uncertainty based steering relation also offers a better prospect of detection of steerability compared to those obtained using the Reid inequality as well as the entropic steering inequality.

In Chapter 4, a sequential detection scheme of genuine tripartite entanglement has been studied. Due to the difficulties present in experimentally producing multipartite quantum correlations, exploring the possibilities of using single multipartite quantum correlation several times is interesting from both foundational as well as information theoretic applications. We consider the scenario where three spin- $\frac{1}{2}$  particles are spatially separated and shared between, say, Alice, Bob and multiple Charlies. Alice and Bob measure on the first and second particle respectively and multiple Charlies measure on the third particle sequentially. In this scenario we investigate how many Charlies can detect genuine entanglement with a single Alice and a single Bob. We have used both linear as well as non-linear correlation inequalities which detect genuine entanglement in the GHZ-state and it has been found that at most two Charlies can detect genuine entanglement of the GHZ-state. Next, using appropriate genuine entanglement witness operators it has been shown that at most four Charlies can detect genuine entanglement sequentially with the single Alice and single Bob using the shared W-state and the number of charlies increases upto twelve when GHZ state is shared among them.

In Chapter 5, a framework for detection of non-Markovianity has been established using uncertainty relations. First, it has been shown that the violation of the Robertson-Schrödinger uncertainty relation for the Choi state of a dynamical evolution is sufficient to detect non-Markovianity. Then some practical examples from quantum channels have been presented to validate the theory. Next, certain linear and non-linear witnesses of non-Markovianity have been proposed. The advantage of the sum uncertainty relation for non-Markovianity detection has also been presented with proper examples. Finally, it has been shown that for unital dynamics non-Markovianity is necessary to decrease the Robertson-Schrödinger uncertainty for the time-evolving physical states. Based on this a new non-Markovianity quantifier is proposed for unital dynamics of qubits.

In Chapter 6, a new quantum communication game called *random-receiver quantum communication* has been introduced. Here, a sender transmits a quantum message to a receiver

chosen from a list of  $n$  spatially separated parties. The choice of receiver is unknown to the sender, but is known by the  $n$  parties who coordinate their actions by exchanging classical messages. In normal conditions, random-receiver quantum communication requires noiseless quantum communication channel from sender to each receivers. In contrast, we show that random-receiver quantum communication can take place through entanglement-breaking channels if the order such channels is controlled by a quantum bit that is accessible through quantum measurements. In particular, the advantage of coherent control of orders in time over that of paths in space is achieved under the distinct role of external and internal degrees of freedoms in communication task.

The studies presented in this thesis lead to the following open problems which are worth for future research works:

- In chapter 3, we proposed a steering-inequality based on the sum uncertainty relation. The proposed result is however for the bipartite scenario. Thus, generalizing our result for multipartite EPR-steering is a direction for future research.
- In Chapter 4, the possibilities of using single genuinely tripartite entangled state several times have been explored in the avenue of sequential measurement scenario. But investigation of direct applications for such sharing protocols [112, 113, 118–124, 129] in information theory and/or cryptography is a matter of future research. Moreover, generalizing the results in the context of more number of parties is another open problem.
- An experimentally feasible detection scheme has been proposed for non-Markovian evolution in chapter 5. The proposed framework however, captures only those non-Markovian evolutions which are CP-indivisible. Investigation of capturing a more generalized description of non-Markovianity (*e.g.*, by process tensor formalism [175]) in an experimentally friendly way is thus a matter of further research. Finally, finding information theoretic applications of non-Markovianity is another area for future research.
- In Chapter 6, random-receiver quantum communication game has been introduced where there is one sender and  $n$  probable receiver. The study opens up potential use of indefinite causal order in distributed protocols, such as multipartite quantum state

transfer, quantum network, and entanglement distribution [248, 249] which have enormous practical relevance in the emerging new technology of quantum internet [250, 251].

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